TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 2, Pages 675–704
S 0002-9947(01)02899-9
Article electronically published on October 3, 2001

TENSOR PRODUCT VARIETIES AND CRYSTALS: GL CASE

ANTON MALKIN

ABSTRACT. A geometric theory of tensor product for $\mathfrak{gl}_N$-crystals is described. In particular, the role of Spaltenstein varieties in the tensor product is explained, and thus a direct (non-combinatorial) proof of the fact that the number of irreducible components of a Spaltenstein variety is equal to a Littlewood-Richardson coefficient (i.e. certain tensor product multiplicity) is obtained.

0. INTRODUCTION

0.1. Given a partition λ of an integer k into N parts (zero parts allowed) one can associate to it various mathematical objects. Three of them are of particular interest to the representation theory: a nilpotent orbit O_λ in $\mathrm{End}_{\mathbb{C}}\,\mathbb{C}^k$ (consisting of operators with the Jordan form given by λ), an irreducible polynomial representation $L(\lambda)$ of GL_N (with the highest weight λ), and an irreducible representation ρ_λ of the symmetric group S_k over $\mathbb{C}$ (defined via the idempotent e_λ in the group algebra $\mathbb{C}S_k$). The goal of the geometric representation theory in this context is to exploit the relation between nilpotent orbits and representations of S_k and GL_N.

0.2. Let $t \in O_\lambda$ (i.e. t is a nilpotent operator in $\mathbb{C}^k$ with Jordan form λ), and let $\mu^1, \dots, \mu^l$ be partitions of integers $k^1, \dots, k^l$ into N parts each. Assume that $k^1 + \dots + k^l = k$, and consider a variety $\mathfrak{S}_l((\mu^1, \dots, \mu^l), \lambda)$ of all l-step partial flags in $\mathbb{C}^k$ with dimensions of the subfactors given by $k^1, \dots, k^l$ and such that t preserves each subspace in the flag, and when restricted to the subfactors it defines operators with the Jordan forms given by $\mu^1, \dots, \mu^l$. $\mathfrak{S}_l((\mu^1, \dots, \mu^l), \lambda)$ is called Spaltenstein variety. It is proven by Spaltenstein (cf. [Spa82, II.5]) that $\mathfrak{S}_l((\mu^1, \dots, \mu^l), \lambda)$ has pure dimension. Let $\mathcal{S}_l((\mu^1, \dots, \mu^l), \lambda)$ be the set of irreducible components of $\mathfrak{S}_l((\mu^1, \dots, \mu^l), \lambda)$. It follows from a theorem of Hall [Hal59] (see also [Mac95, Chapter II]) that the cardinal of $\mathcal{S}_l((\mu^1, \dots, \mu^l), \lambda)$ (i.e. the number of irreducible components of $\mathfrak{S}_l((\mu^1, \dots, \mu^l), \lambda)$ is equal to the multiplicity of the representation L_λ of GL_N in the tensor product $L_{\mu^1} \otimes \dots \otimes L_{\mu^l}$:

$$|\mathcal{S}_l((\mu^1, \dots, \mu^l), \lambda)| = \dim_{\mathbb{C}} \mathrm{Hom}_{GL_N}(L_{\mu^1} \otimes \dots \otimes L_{\mu^l},\ L_\lambda)\ , \tag{0.2.a}$$

or to the multiplicity of $\rho_{\mu^1} \otimes \dots \otimes \rho_{\mu^l}$ in the restriction of ρ_λ to $S_{k^1} \times \dots \times S_{k^l}$:

(0.2.b)
$$|\mathcal{S}_l((\mu^1, \dots, \mu^l), \lambda)| = \dim_{\mathbb{C}} \mathrm{Hom}_{S_{k^1} \times \dots \times S_{k^l}}(\rho_{\mu^1} \otimes \dots \otimes \rho_{\mu^l},\ \mathrm{res}_{S_k}^{S_{k^1} \times \dots \times S_{k^l}} \rho_\lambda)\ .$$

Received by the editors March 7, 2001.
2000 *Mathematics Subject Classification.* Primary 20G99, 14M15.

The right hand sides of (0.2.a) and (0.2.b) are equal due to Schur-Weyl duality, and either equality can be derived by combinatorial means (for example from the Littlewood-Richardson Rule – cf. [Mac95, Chapter II]). However it is interesting to understand the geometric meaning of (0.2.a) and (0.2.b) (i.e. to find the role of the variety $\mathfrak{S}_l((\mu^1,\dots,\mu^l),\lambda)$ in the tensor product for GL_N and in the restriction functor for symmetric groups).

The statement (0.2.b) was proven geometrically by Borho and MacPherson [BM83] using Springer's realization of representations of symmetric groups.

The goal of this paper is to develop a geometric theory of tensor product for polynomial representations of GL_N, or, more exactly, for gl_N-crystals. In this geometric setup the variety $\mathfrak{S}_l((\mu^1,\dots,\mu^l),\lambda)$ is responsible for the multiplicities in the tensor product decomposition. In particular, (0.2.a) is an immediate corollary of the main theorem of the paper (Theorem 3.9).

0.3. Let t be again a nilpotent operator in $\mathbb{C}^k$ with Jordan form λ (i.e. $t \in O_\lambda$). Consider the variety $\mathfrak{M}_N(\lambda)$ of all N-step partial flags $(\{0\} = F_0 \subset F_1 \subset \ldots \subset F_N = \mathbb{C}^k)$ in $\mathbb{C}^k$ such that $tF_k \subset F_{k-1}$ for any $k = 1,\dots,N$. Note that the dimensions of the subfactors of the flag are not fixed. Each connected component of the variety $\mathfrak{M}_N(\lambda)$ is of pure dimension. Ginzburg [Gin91] constructed a representation of GL_N in the top homology of $\mathfrak{M}_N(\lambda)$ (more exactly, in the sum of the top Borel-Moore homology groups of connected components of $\mathfrak{M}_N(\lambda)$), and proved that this representation is isomorphic to $L(\lambda)$. This paper deals with a weaker structure – that of Kashiwara's gl_N-crystal on the set $\mathcal{M}_N(\lambda)$ of irreducible components of $\mathfrak{M}_N(\lambda)$. This crystal is isomorphic to the crystal $\mathcal{L}(\lambda)$ of the canonical basis of $L(\lambda)$.

0.4. The main idea of this paper is to construct a certain variety $\mathfrak{T}_N(\mu^1,\dots,\mu^l)$ such that its connected components are of pure dimension and the set $\mathcal{T}_N(\mu^1,\dots,\mu^l)$ of its irreducible components can be equipped with a structure of gl_N-crystal in such a way that one has two crystal isomorphisms:

$$\mathcal{T}_N(\mu^1,\dots,\mu^l) \approx \mathcal{M}_N(\mu^1) \otimes \ldots \otimes \mathcal{M}_N(\mu^l) \tag{0.4.a}$$

and

$$\mathcal{T}_N(\mu^1,\dots,\mu^l) \approx \bigoplus_{\lambda} \mathcal{S}_l((\mu^1,\dots,\mu^l),\lambda) \otimes \mathcal{M}_N(\lambda)\ , \tag{0.4.b}$$

where the set $\mathcal{S}_l((\mu^1,\dots,\mu^l),\lambda)$ is considered as a trivial crystal, and $\otimes$ (resp. $\oplus$) denotes the tensor product (resp. direct sum) of crystals, which is equal to the direct product (resp. disjoint union) on the level of sets.

The variety $\mathfrak{T}_N(\mu^1,\dots,\mu^l)$ is defined as the variety of all triples $(t,\mathbf{X},\mathbf{F})$ consisting of a nilpotent operator t in $\operatorname{End}_{\mathbb{C}}\mathbb{C}^k$, an l-step partial flag $\mathbf{X}$ in $\mathbb{C}^k$, and an N-step partial flag $\mathbf{F}$ in $\mathbb{C}^k$, such that subspaces of both flags are preserved by t, and when t is restricted to the subfactors of $\mathbf{X}$ (resp. $\mathbf{F}$) it gives operators with the Jordan forms $\mu^1,\dots,\mu^l$ (resp. 0 operators). In particular, the dimensions of the subfactors of $\mathbf{X}$ are equal to $k^1,\dots,k^l$, but the dimensions of the subfactors of $\mathbf{F}$ can vary. The variety $\mathfrak{T}_N(\mu^1,\dots,\mu^l)$ is called tensor product variety.

To describe the bijection (0.4.a) restrict the flag $\mathbf{F}$ onto the subfactors of the flag $\mathbf{X}$. In this way one obtains a map from $\mathfrak{T}_N(\mu^1,\dots,\mu^l)$ to $\mathfrak{M}_N(\mu^1) \times \ldots \times \mathfrak{M}_N(\mu^l)$, which induces the bijection (0.4.a).

To obtain the bijection (0.4.b) consider $\mathfrak{T}_N(\mu^1,\dots,\mu^l)$ as a fibration over the nilpotent cone in $\mathrm{End}_{\mathbb{C}}\mathbb{C}^k$ (the projection map takes $(t,\mathbf{X},\mathbf{F})$ to t). The fiber of this fibration over a point $t\in O_\lambda$ is isomorphic to the product of varieties $\mathfrak{S}_l((\mu^1,\dots,\mu^l),\lambda)$ and $\mathfrak{M}_N(\lambda)$. Moreover, the dimension of the preimage of O_λ in a connected component of $\mathfrak{T}_N(\mu^1,\dots,\mu^l)$ does not depend on λ. Hence one has the bijection (0.4.b).

0.5. The most important result of the paper is that the crystal structures on $\mathcal{T}_N(\mu^1,\dots,\mu^l)$ induced by the bijections (0.4.a) and (0.4.b) are the same, in other words that the composite bijection

$$\text{(0.5.a)}\qquad \tau_N:\quad \mathcal{M}_N(\mu^1)\otimes\ldots\otimes\mathcal{M}_N(\mu^l)\xrightarrow{\sim}\bigoplus_\lambda \mathcal{S}_l((\mu^1,\dots,\mu^l),\lambda)\otimes\mathcal{M}_N(\lambda)$$

is a crystal isomorphism. This is shown by reducing the problem to the GL_2 case.

Actually, in the main body of the paper only the case of two multiples (i.e. $l=2$) is considered. However generalization to arbitrary l is straightforward.

0.6. Certainly, another proof of (0.2.a) would not be worth the effort. However the understanding of its geometric meaning paves the way for generalizations to Kac-Moody algebras other than gl_N. In particular, in [Mal01] the ADE case is considered, and certain "multiplicity variety" is defined such that the number of its irreducible components is equal to the multiplicity of a simple finite dimensional representation of a Lie algebra of ADE type in the tensor product of l simple finite dimensional representations. The multiplicity variety is a generalization of the Spaltenstein variety in the context of Nakajima's theory of quiver varieties.

0.7. The paper is organized as follows. Section 1 contains a short review of Kashiwara's theory of crystals. In Section 2 the geometric tensor product is described in the GL_2 case. This demonstrates general constructions on a simple example, but a more important reason for including this section is that the generic GL_N case is reduced later to $N=2$. In Section 3 the case of GL_N is considered. Both Sections 2 and 3 follow the same pattern. First, tensor product and Spaltenstein varieties are defined, then the bijections α and β are described using certain "tensor product" diagrams, and finally it is proven that the composite bijection τ is a crystal morphism (in the case of $N=2$ by a direct verification and for arbitrary N via reduction to gl_2-subalgebras inside gl_N).

Most of the proofs consist of tedious checks of dimensions of various locally closed subsets of the varieties involved. The only non-routine result is (2.5.g) which is the reason for τ_N to be a crystal morphism.

0.8. The tensor product variety in the special case $\mu^1=(k^1,0,0,\dots),\ \dots,\ \mu^l=(k^l,0,0,\dots)$, was independently described by H. Nakajima [Nak01, 9.1].

In the case when $k^1=\ldots=k^l=1$ (i.e. the case of the product of l copies of the fundamental representation of GL_N) the tensor product variety is a certain Lagrangian subvariety of the cotangent bundle to the variety considered by Grojnowski and Lusztig in [GL92].

0.9. Throughout the paper the following conventions are used: the ground field is $\mathbb{C}$; "closed", "locally closed", etc. refer to the Zariski topology; in the phrase "locally trivial fibration" "locally" refers to the Zariski topology, however trivialization is analytic (not regular).

1. Crystals

Crystals were unearthed by Kashiwara [Kas90, Kas91, Kas94]. An excellent survey of crystals as well as some new results are given by Joseph in [Jos95, Chapters 5, 6].

1.1. **Definition of $\mathfrak{g}$-crystals.** Let $\mathfrak{g}$ be a reductive or a Kac-Moody Lie algebra, I be the set of vertices of the Dynkin graph of $\mathfrak{g}$, $\mathcal{Q}_{\mathfrak{g}}$ (resp. $\mathcal{Q}_{\mathfrak{g}}^{\vee}$) be the weight (resp. coweight) lattice of $\mathfrak{g}$, $\{\alpha_i \in \mathcal{Q}_{\mathfrak{g}}\}_{i\in I}$ (resp. $\{\alpha_i^{\vee} \in \mathcal{Q}_{\mathfrak{g}}^{\vee}\}_{i\in I}$) be the set of simple roots (resp. simple coroots), $\langle , \rangle$ be the natural pairing between $\mathcal{Q}_{\mathfrak{g}}$ and $\mathcal{Q}_{\mathfrak{g}}^{\vee}$.

A $\mathfrak{g}$-*crystal* is a tuple $(\mathcal{A}, wt, \{\varepsilon_i\}_{i\in I}, \{\varphi_i\}_{i\in I}, \{\tilde{e}_i\}_{i\in I}, \{\tilde{f}_i\}_{i\in I})$, where

- $\mathcal{A}$ is a set,
- wt is a map from $\mathcal{A}$ to $\mathcal{Q}_{\mathfrak{g}}$,
- ε_i and φ_i are maps from $\mathcal{A}$ to $\mathbb{Z}$,
- $\tilde{e}_i$ and $\tilde{f}_i$ are maps from $\mathcal{A}$ to $\mathcal{A} \cup \{0\}$.

These data should satisfy the following axioms:

- $wt(\tilde{e}_i a) = wt(a) + \alpha_i, \quad \varphi_i(\tilde{e}_i a) = \varphi_i(a) + 1, \quad \varepsilon_i(\tilde{e}_i a) = \varepsilon_i(a) - 1,$ for any $i \in I$ and $a \in \mathcal{A}$ such that $\tilde{e}_i a \neq 0$,
- $wt(\tilde{f}_i a) = wt(a) - \alpha_i, \quad \varphi_i(\tilde{f}_i a) = \varphi_i(a) - 1, \quad \varepsilon_i(\tilde{f}_i a) = \varepsilon_i(a) + 1,$ for any $i \in I$ and $a \in \mathcal{A}$ such that $\tilde{f}_i a \neq 0$,
- $\langle \alpha_i^{\vee}, wt(a)\rangle = \varphi_i(a) - \varepsilon_i(a)$ for any $i \in I$ and $a \in \mathcal{A}$,
- $\tilde{f}_i a = b$ if and only if $\tilde{e}_i b = a$, where $i \in I$, and $a, b \in \mathcal{A}$.

The maps $\tilde{e}_i$ and $\tilde{f}_i$ are called Kashiwara's operators, and the map wt is called the weight function.

Remark. In a more general definition of crystals the maps ε_i and φ_i are allowed to have infinite values.

A $\mathfrak{g}$-crystal $(\mathcal{A}, wt, \{\varepsilon_i\}_{i\in I}, \{\varphi_i\}_{i\in I}, \{\tilde{e}_i\}_{i\in I}, \{\tilde{f}_i\}_{i\in I})$ is called *trivial* if

$$wt(a) = 0 \text{ for any } a \in \mathcal{A},$$
$$\varepsilon_i(a) = \varphi_i(a) = 0 \text{ for any } i \in I \text{ and } a \in \mathcal{A},$$
$$\tilde{e}_i a = \tilde{f}_i a = 0 \text{ for any } i \in I \text{ and } a \in \mathcal{A}.$$

Any set $\mathcal{A}$ can be equipped with the trivial crystal structure as above.

A $\mathfrak{g}$-crystal $(\mathcal{A}, wt, \{\varepsilon_i\}_{i\in I}, \{\varphi_i\}_{i\in I}, \{\tilde{e}_i\}_{i\in I}, \{\tilde{f}_i\}_{i\in I})$ is called *normal* if

$$\varepsilon_i(a) = \max\{n \mid \tilde{e}_i^n a \neq 0\}\ ,$$
$$\varphi_i(a) = \max\{n \mid \tilde{f}_i^n a \neq 0\}\ ,$$

for any $i \in I$ and $a \in \mathcal{A}$. Thus in a normal $\mathfrak{g}$-crystal the maps ε_i and φ_i are uniquely determined by the action of $\tilde{e}_i$ and $\tilde{f}_i$. A trivial crystal is normal.

In the rest of the paper all $\mathfrak{g}$-crystals are assumed normal, and thus the maps ε_i and φ_i are usually omitted.

By abuse of notation a $\mathfrak{g}$-crystal $(\mathcal{A}, wt, \{\varepsilon_i\}_{i\in I}, \{\varphi_i\}_{i\in I}, \{\tilde{e}_i\}_{i\in I}, \{\tilde{f}_i\}_{i\in I})$ is sometimes denoted simply by $\mathcal{A}$.

An *isomorphism* of two $\mathfrak{g}$-crystals $\mathcal{A}$ and $\mathcal{B}$ is a bijection between the sets $\mathcal{A}$ and $\mathcal{B}$ commuting with the action of the operators $\tilde{e}_i$ and $\tilde{f}_i$, and the functions wt, ε_i, and φ_i.

The *direct sum* $\mathcal{A}\oplus\mathcal{B}$ of two $\mathfrak{g}$-crystals $\mathcal{A}$ and $\mathcal{B}$ is their disjoint union as sets with the maps $\tilde{e}_i$, $\tilde{f}_i$, wt, ε_i, and φ_i acting on each component of the union separately.

1.2. **Tensor product of $\mathfrak{g}$-crystals.** The *tensor product* $\mathcal{A}\otimes\mathcal{B}$ of two $\mathfrak{g}$-crystals $\mathcal{A}$ and $\mathcal{B}$ is their direct product as sets equipped with the following crystal structure:

$$wt((a,b)) = wt(a) + wt(b) ,$$
$$\varepsilon_i((a,b)) = \max\{\varepsilon_i(a), \varepsilon_i(a) + \varepsilon_i(b) - \varphi_i(a)\} ,$$
$$\varphi_i((a,b)) = \max\{\varphi_i(b), \varphi_i(a) + \varphi_i(b) - \varepsilon_i(b)\} ,$$

(1.2.a)
$$\tilde{e}_i((a,b)) = \begin{cases} (\tilde{e}_i a \ , \ b) & \text{if } \varphi_i(a) \geq \varepsilon_i(b) , \\ (a \ , \ \tilde{e}_i b) & \text{if } \varphi_i(a) < \varepsilon_i(b) , \end{cases}$$
$$\tilde{f}_i((a,b)) = \begin{cases} (\tilde{f}_i a \ , \ b) & \text{if } \varphi_i(a) > \varepsilon_i(b) , \\ (a \ , \ \tilde{f}_i b) & \text{if } \varphi_i(a) \leq \varepsilon_i(b) . \end{cases}$$

Here $(a, 0)$, $(0, b)$, and $(0, 0)$ are identified with 0. One can check that the set $\mathcal{A}\times\mathcal{B}$ with the above structure satisfies all the axioms of a (normal) crystal, and that the tensor product of crystals is associative.

Remark. Tensor product of crystals is not commutative.

Remark. The tensor product (in any order) of a crystal $\mathcal{A}$ with a trivial crystal $\mathcal{B}$ is isomorphic to the direct sum $\bigoplus_{b\in\mathcal{B}} \mathcal{A}_b$ where each $\mathcal{A}_b$ is isomorphic to $\mathcal{A}$.

1.3. **Highest weight crystals and closed families.** A crystal $\mathcal{A}$ is a *highest weight* crystal with the highest weight $\lambda \in \mathcal{Q}_\mathfrak{g}$ if there exists an element $a_\lambda \in \mathcal{A}$ such that:

- $wt(a_\lambda) = \lambda$,
- $\tilde{e}_i a_\lambda = 0$ for any $i \in I$,
- any element of $\mathcal{A}$ can be obtained from a_λ by successive applications of the operators $\tilde{f}_i$.

Consider a family of highest weight normal crystals $\{\mathcal{A}(\lambda)\}_{\lambda\in\mathcal{J}}$ labeled by a set $\mathcal{J} \subset \mathcal{Q}_\mathfrak{g}$ (the highest weight of $\mathcal{A}(\lambda)$ is λ). The family $\{\mathcal{A}(\lambda)\}_{\lambda\in\mathcal{J}}$ is called *strictly closed* (with respect to tensor products) if the tensor product of any two members of the family is isomorphic to a direct sum of members of the family:

$$\mathcal{A}(\mu^1) \otimes \mathcal{A}(\mu^2) = \bigoplus_{\lambda\in\mathcal{J}} \mathcal{U}((\mu^1, \mu^2), \lambda) \otimes \mathcal{A}(\lambda) ,$$

where $\mathcal{U}((\mu^1, \mu^2), \lambda)$ are sets with the trivial crystal structures. A family $\{\mathcal{A}(\lambda)\}_{\lambda\in\mathcal{J}}$ is called *closed* if the tensor product $\mathcal{A}(\mu^1)\otimes\mathcal{A}(\mu^2)$ of any two members of the family contains $\mathcal{A}(\mu^1 + \mu^2)$ as a direct summand. Any strictly closed family is closed.

Let $\mathcal{Q}^+_\mathfrak{g} \subset \mathcal{Q}_\mathfrak{g}$ be the set of highest weights of integrable highest weights modules of $\mathfrak{g}$ (in the reductive case a module is called integrable if it is derived from a polynomial representation of the corresponding connected simply connected reductive group, in the Kac-Moody case the highest weight should be a positive linear combination of the fundamental weights). The original motivation for the introduction of crystals was the discovery by Kashiwara [Kas91] and Lusztig [Lus91] of canonical (or crystal) bases in integrable highest weight modules of a (quantum) Kac-Moody algebra. These bases have many favorable properties, one of which is that as sets they are equipped with a crystal structure. In other words, to each irreducible integrable highest weight module $L(\lambda)$ corresponds a normal crystal $\mathcal{L}(\lambda)$ (the crystal of the canonical basis). In this way one obtains a strictly closed family of crystals $\{\mathcal{L}(\lambda)\}_{\lambda\in\mathcal{Q}^+_\mathfrak{g}}$, satisfying the following two properties:

- the cardinal of $\mathcal{L}(\lambda)$ is equal to the dimension of $L(\lambda)$;
- one has the following tensor product decompositions for $\mathfrak{g}$-modules and $\mathfrak{g}$-crystals:
$$L(\mu^1) \otimes L(\mu^2) = \bigoplus_{\lambda \in \mathcal{Q}_{\mathfrak{g}}^+} C((\mu^1, \mu^2), \lambda) \otimes L(\lambda) \ ,$$
$$\mathcal{L}(\mu^1) \otimes \mathcal{L}(\mu^2) = \bigoplus_{\lambda \in \mathcal{Q}_{\mathfrak{g}}^+} \mathcal{C}((\mu^1, \mu^2), \lambda) \otimes \mathcal{L}(\lambda) \ ,$$
where the cardinal of the set (trivial crystal) $\mathcal{C}((\mu^1, \mu^2), \lambda)$ is equal to the dimension of the linear space (trivial $\mathfrak{g}$-module) $C((\mu^1, \mu^2), \lambda)$.

The aim of this paper is to construct another strictly closed family of $\mathfrak{g}$-crystals $\{\mathcal{M}(\lambda)\}_{\lambda \in \mathcal{Q}_{\mathfrak{g}}^+}$ for $\mathfrak{g} = gl_N$ using geometry associated to $\mathfrak{g}$. The following crucial theorem ensures that this family is isomorphic to the family $\{\mathcal{L}(\lambda)\}_{\lambda \in \mathcal{Q}_{\mathfrak{g}}^+}$ of crystals of canonical bases (two families of crystals labeled by the same index set are called isomorphic if the corresponding members of the families are isomorphic as crystals).

Theorem. *There exists a unique (up to an isomorphism) closed family of $\mathfrak{g}$-crystals labeled by $\mathcal{Q}_{\mathfrak{g}}^+$.*

Proof. For a proof of the Kac-Moody case see [Jos95, Proposition 6.4.21]. The statement for reductive $\mathfrak{g}$ follows easily from the statement for the factor of $\mathfrak{g}$ by its center. □

2. Grassmann varieties and gl_2-crystals

This section contains a geometric description of a closed family of gl_2-crystals.

2.1. **Notation.** From now on the weight lattice $\mathcal{Q}_{gl_2}$ of gl_2 is denoted simply by $\mathcal{Q}_2$ and is identified with $\mathbb{Z} \oplus \mathbb{Z}$.

Given $n, m \in \mathbb{Z}$, Gr^n_m denotes the Grassmann variety of all m-dimensional subspaces of $\mathbb{C}^n$. If $m < 0$ or $n < m$ then Gr^n_m is empty. Otherwise it is a smooth connected variety of dimension $m(n-m)$. Let $Gr(n) = \bigsqcup_{0 \le m \le n} Gr^n_m$ be the variety of all subspaces of $\mathbb{C}^n$.

2.2. **Geometric gl_2-crystals.** Given $w \in \mathbb{Z}_{\geq 0}$ consider the following diagram of varieties (cf. [Gin91]):

(2.2.a)
$$\begin{array}{c} \mathfrak{M}_2(w) \\ \downarrow{\scriptstyle \pi_2} \\ \mathfrak{N}_2(w) \ , \end{array}$$

where

$\mathfrak{N}_2(w) = \{t \in \mathrm{End}(\mathbb{C}^w) \mid t^2 = 0\}$,

$\mathfrak{M}_2(w) = \{(t, F) \mid t \in \mathfrak{N}_2(w),\ F \in Gr(w),\ \mathrm{im}\, t \subset F \subset \ker t\}$, in other words, $\mathfrak{M}_2(w)$ is a variety of pairs (t, F) consisting of an operator t in $\mathbb{C}^w$ and a subspace $F \subset \mathbb{C}^w$, such that $\mathrm{im}\, t \subset F \subset \ker t$,

$\pi_2((t, F)) = t$.

The variety $\mathfrak{M}_2(w)$ is a disjoint union of connected components:
$$\mathfrak{M}_2(w) = \bigsqcup_{v \in \mathbb{Z}_{\geq 0}} \mathfrak{M}_2(v, w) \ ,$$

where

$$\mathfrak{M}_2(v,w) = \{(t,F) \in \mathfrak{M}_2(w) \mid \dim F = v\} \; .$$

The variety $\mathfrak{N}_2(w)$ can be stratified as follows:

$$\mathfrak{N}_2(w) = \bigcup_{r \in \mathbb{Z}_{\geq 0}} O_{(w-r,r)} \; ,$$

where

$$O_{(w-r,r)} = \{t \in \mathfrak{N}_2(w) \mid \operatorname{rank} t = r\} \; .$$

The weird way of writing index λ of O_λ will be explained in subsection 3.2. Each stratum O_λ is a single $GL(w)$-orbit in $\mathfrak{N}_2(w)$ and there are finitely many non-empty strata. The dimension of $O_{(w-r,r)}$ is given by

$$\dim O_{(w-r,r)} = 2r(w-r) \; . \tag{2.2.b}$$

Choose $t \in O_{(w-r,r)}$ and consider the variety $\mathfrak{M}_2(w,r) = \pi_2^{-1}(t)$. This variety does not depend (up to an isomorphism) on the choice of $t \in O_{(w-r,r)}$. However if a specific t is used then the notation is $\mathfrak{M}_2(w,t)$ instead of $\mathfrak{M}_2(w,r)$.

The variety $\mathfrak{M}_2(w,r)$ is a disjoint union of connected components:

$$\mathfrak{M}_2(w,r) = \bigsqcup_{v \in \mathbb{Z}_{\geq 0}} \mathfrak{M}_2(v,w,r) \; ,$$

where

$$\mathfrak{M}_2(v,w,r) = \mathfrak{M}_2(w,r) \cap \mathfrak{M}_2(v,w) \; .$$

The component $\mathfrak{M}_2(v,w,r)$ is isomorphic to Gr_{v-r}^{w-2r} (in particular it is nonempty only for a finite number of values of v). Thus $\mathfrak{M}_2(v,w,r)$ is a smooth connected variety of dimension

$$\dim \mathfrak{M}_2(v,w,r) = (w-r-v)(v-r) \; . \tag{2.2.c}$$

Let $\mathcal{M}_2(w,r) = \{\mathfrak{M}_2(v,w,r)\}_{v=r}^{w-r}$ be the set of irreducible components of $\mathfrak{M}_2(w,r)$. Endow this set with the following structure of a gl_2-crystal:

$$\begin{aligned}
wt(\mathfrak{M}_2(v,w,r)) &= (v, w-v) \; , \\
\varepsilon(\mathfrak{M}_2(v,w,r)) &= w-r-v \; , \\
\varphi(\mathfrak{M}_2(v,w,r)) &= v-r \; , \\
\tilde{e}(\mathfrak{M}_2(v,w,r)) &= \begin{cases} \mathfrak{M}_2(v+1,w,r) & \text{if } v < w-r \; , \\ 0 & \text{if } v \geq w-r \; , \end{cases} \\
\tilde{f}(\mathfrak{M}_2(v,w,r)) &= \begin{cases} \mathfrak{M}_2(v-1,w,r) & \text{if } v > r \; , \\ 0 & \text{if } v \leq r \; . \end{cases}
\end{aligned} \tag{2.2.d}$$

Here indexes of ε, φ, $\tilde{e}$, and $\tilde{f}$ are omitted because gl_2 has only one simple root. The set $\mathcal{M}_2(w,r)$ equipped with this structure is a highest weight normal gl_2-crystal with the highest weight $(w-r,r)$. In other words, it is isomorphic (as a crystal) to $\mathcal{L}((w-r,r))$.

The above is a trivial example of a geometric construction of crystals due to Lusztig and Nakajima (cf. [Lus91, Nak98, KS97]). The next subsection provides a geometric description of the tensor product on the family $\{\mathcal{M}_2(w,r)\}_{w,r \in \mathbb{Z}_{\geq 0}}$.

2.3. **The tensor product variety.** Let w^1, w^2, r^1, r^2 be non-negative integers, and let $\mathfrak{T}_2(w^1, r^1, w^2, r^2)$ be a variety of triples (t, X, F), where

$t \in \mathfrak{N}_2(w^1 + w^2)$;

F is a subspace of $\mathbb{C}^{w^1+w^2}$, such that $\operatorname{im} t \subset F \subset \ker t$ (i.e. $(t, F) \in \mathfrak{M}_2(w^1+w^2)$);

X is a subspace of $\mathbb{C}^{w^1+w^2}$ such that $\dim X = w^1$, $tX \subset X$, $\operatorname{rank}(t|_X) = r^1$, and $\operatorname{rank}(t|_{(\mathbb{C}^{w^1+w^2}/X)}) = r^2$.

The variety $\mathfrak{T}_2(w^1, r^1, w^2, r^2)$ is called *tensor product variety* for its role in the description of the gl_2-crystal $\mathcal{M}_2(w^1, r^1) \otimes \mathcal{M}_2(w^2, r^2)$. One has the following decomposition of $\mathfrak{T}_2(w^1, r^1, w^2, r^2)$ into a disjoint union of varieties:

$$\mathfrak{T}_2(w^1, r^1, w^2, r^2) = \bigsqcup_{v \in \mathbb{Z}_{\geq 0}} \mathfrak{T}_2(v, w^1, r^1, w^2, r^2) ,$$

where

$$\mathfrak{T}_2(v, w^1, r^1, w^2, r^2) = \{(t, X, F) \in \mathfrak{T}_2(w^1, r^1, w^2, r^2) \mid \dim F = v\} .$$

Moreover

$$\mathfrak{T}_2(w^1, r^1, w^2, r^2) = \bigcup_{v^1, v^2 \in \mathbb{Z}_{\geq 0}} \mathfrak{T}_2(v^1, w^1, r^1, v^2, w^2, r^2) ,$$

where

$$\begin{aligned} &\mathfrak{T}_2(v^1, w^1, r^1, v^2, w^2, r^2) \\ &\quad = \{(t, X, F) \in \mathfrak{T}_2(w^1, r^1, w^2, r^2) \mid \dim F \cap X = v^1,\ \dim F/(F \cap X) = v^2\} . \end{aligned}$$

Each subset $\mathfrak{T}_2(v^1, w^1, r^1, v^2, w^2, r^2)$ is locally closed and one has

$$\mathfrak{T}_2(v, w^1, r^1, w^2, r^2) = \bigcup_{\substack{v^1, v^2 \in \mathbb{Z}_{\geq 0} \\ v^1 + v^2 = v}} \mathfrak{T}_2(v^1, w^1, r^1, v^2, w^2, r^2) .$$

In the next several subsections the variety $\mathfrak{T}_2(w^1, r^1, w^2, r^2)$ is used to describe the tensor product of gl_2-crystals $\mathcal{M}_2(w^1, r^1)$ and $\mathcal{M}_2(w^2, r^2)$. The set of irreducible components of $\mathfrak{T}_2(w^1, r^1, w^2, r^2)$ is seen to be in bijection with the set $\mathcal{M}_2(w^1, r^1) \times \mathcal{M}_2(w^2, r^2)$, and two ways of labeling the irreducible components imply the decomposition of the tensor product as a direct sum.

2.4. **The tensor product variety as a fibered product.** Recall that the variety $\mathfrak{N}_2(w^1 + w^2)$ can be stratified by $GL(w^1 + w^2)$-orbits (cf. subsection 2.2):

$$\mathfrak{N}_2(w^1 + w^2) = \bigcup_{r \in \mathbb{Z}_{\geq 0}} O_{(w^1+w^2-r, r)} ,$$

where

$$O_{(w^1+w^2-r, r)} = \{t \in \mathfrak{N}_2(w^1 + w^2) \mid \operatorname{rank} t = r\} .$$

The dimension of $O_{(w^1+w^2-r, r)}$ is given by (2.2.b):

$$\dim O_{(w^1+w^2-r, r)} = 2r(w^1 + w^2 - r) . \tag{2.4.a}$$

Consider a diagram

$$\begin{array}{c} \mathfrak{T}_2(v,w^1,r^1,w^2,r^2) \\ \Big\downarrow{\scriptstyle \delta_2} \\ \mathfrak{N}_2(w^1+w^2)\ , \end{array}$$

where $\delta_2((t,X,F)) = t$. The map δ_2 restricted to $\delta_2^{-1}(O_{(w^1+w^2-r,r)})$ is a locally trivial fibration with a fiber isomorphic to a direct product

$$\mathfrak{M}_2(v,w^1+w^2,r)\times\mathfrak{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))\ ,$$

where the variety $\mathfrak{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))$ can be described as follows. Fix $t\in O_{(w^1+w^2-r,r)}$. Then $\mathfrak{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))$ is a subvariety of the Grassmannian of w^1-dimensional subspaces of $\mathbb{C}^{w^1+w^2}$ given by

$$\begin{aligned}\mathfrak{S}_2&(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))\\ &=\{X\subset\mathbb{C}^{w^1+w^2}\mid \dim X=w^1,\ tX\subset X,\\ &\qquad \operatorname{rank}(t|_X)=r^1,\ \operatorname{rank}(t|_{\mathbb{C}^{w^1+w^2}/X})=r^2\}.\end{aligned}$$

The variety $\mathfrak{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))$ is an example of a Spaltenstein variety (cf. subsection 3.2). It does not depend (up to an isomorphism) on the choice of $t\in O_{(w^1+w^2-r,r)}$.

Proposition. *If the inequality*

$$r^1+r^2\le r\le\min(w^2-r^2+r^1,w^1-r^1+r^2)$$

holds, then the variety $\mathfrak{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))$ *is a smooth connected quasi-projective variety of dimension*

$$\begin{aligned}\dim&\,\mathfrak{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))\\ &=w^1w^2+\frac{1}{2}(\dim O_{(w^1-r^1,r^1)}+\dim O_{(w^2-r^2,r^2)}-\dim O_{(w^1+w^2-r,r)})\ .\end{aligned}$$

Otherwise $\mathfrak{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))$ *is empty.*

Proof. Fix $t\in\mathfrak{N}_2(w^1+w^2)$, $\operatorname{rank}t=r$. The operator t defines a flag

$$\{0\}\subset\operatorname{im}t\subset\ker t\subset\mathbb{C}^{w^1+w^2}\ .$$

The variety $\mathfrak{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))$ is a variety of all linear subspaces $X\subset\mathbb{C}^{w^1+w^2}$ such that

$$\begin{gathered}\dim(X\cap\operatorname{im}t)=r-r^2,\\ \operatorname{im}(t|_X)\subset(X\cap\operatorname{im}t),\\ \dim(\operatorname{im}t|_X)=r^1,\\ \dim((X\cap\ker t)/(X\cap\operatorname{im}t))=w^1-r-r^1+r^2.\end{gathered}$$

The idea of the proof of the proposition is to construct a subspace X satisfying the above conditions in several steps. On each step the choices are parametrized by a certain variety. Start with $X\cap\operatorname{im}t$. It is an arbitrary subspace of dimension $r-r^2$ in $\operatorname{im}t$. So one has a Grassmannian $Gr^r_{r-r^2}$ of such subspaces. Having fixed $X\cap\operatorname{im}t$ choose a subspace $\operatorname{im}(t|_X)$ inside of it (the choices are parametrized by the Grassmannian $Gr_{r^1}^{r-r^2}$). The next step is to select $X \cap \ker t$. The intersection $X \cap \operatorname{im} t$ is already fixed. Thus one proceeds in two steps. Choose $(X \cap \ker t)/(X \cap \operatorname{im} t)$ inside $\ker t/\operatorname{im} t$ (which gives $Gr_{w^1-r-r^1+r^2}^{w^1+w^2-2r}$). Then each choice of $X \cap \ker t$ corresponds to an element of $\operatorname{Hom}((X \cap \ker t)/(X \cap \operatorname{im} t), (\operatorname{im} t/(X \cap \operatorname{im} t)))$. At this point $(X \cap \ker t) \subset \ker t$ and $(X/(X \cap \ker t)) \subset \mathbb{C}^{w^1+w^2}/\ker t$ are fixed (the latter is determined by $\operatorname{im}(t|_X)$). Thus the remaining choices for X correspond to elements of $\operatorname{Hom}((X/(X \cap \ker t)), (\ker t/(X \cap \ker t)))$.

A rigorous way to spell out the above argument is to say that one has the following chain of locally trivial fibrations

$$pt \leftarrow A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow A_4 \leftarrow \mathfrak{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r))$$

with fibers isomorphic to $Gr_{r-r^2}^{r}$, $Gr_{r^1}^{r-r^2}$, $Gr_{w^1-r-r^1+r^2}^{w^1+w^2-2r}$, $\operatorname{Hom}(\mathbb{C}^{w^1-r-r^1+r^2}, \mathbb{C}^{r^2})$, and $\operatorname{Hom}(\mathbb{C}^{r^1}, \mathbb{C}^{w^2-r+r^1})$ respectively.

The proposition follows. $\square$

The statement about the dimension of $\mathfrak{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r))$ is a special case of the Spaltenstein theorem (cf. (3.2)).

Let $\mathcal{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r))$ denote the set of irreducible components of the variety $\mathfrak{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r))$. Thus

$$\text{(2.4.b)} \quad \mathcal{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r)) \\ = \begin{cases} \{\mathfrak{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r))\} & \text{if } r \geq r^1 + r^2 \text{ and} \\ & r \leq \min\{w^2 - r^2 + r^1, w^1 - r^1 + r^2\}, \\ \emptyset & \text{otherwise .} \end{cases}$$

This description of the set $\mathcal{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r))$ of irreducible components of $\mathfrak{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r))$ as a one-element or the empty set is a special case of the Hall theorem (cf. (3.3)).

Turning back to the variety $\mathfrak{T}_2(w^1, r^1, w^2, r^2)$ recall that the fiber of the map

$$\delta_2 : \mathfrak{T}_2(v, w^1, r^1, w^2, r^2) \to \mathfrak{N}_2(w^1 + w^2)$$

over a point in $O_{(w^1+w^2-r,r)}$ is isomorphic to

$$\mathfrak{M}_2(v, w^1 + w^2, r) \times \mathfrak{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r)) \ .$$

Proposition 2.4 and formulas (2.2.c) and (2.4.a) imply that $\delta_2^{-1}(O_{(w^1+w^2-r,r)})$ is empty or a smooth connected quasi-projective variety of dimension

$$\dim \delta_2^{-1}(O_{(w^1+w^2-r,r)}) \\ = w^1 w^2 + v(w^1 + w^2 - v) + \frac{1}{2}(\dim O_{(w^1-r^1,r^1)} + \dim O_{(w^2-r^2,r^2)}) \ .$$

In particular the dimension does not depend on r (or, in other words, on the stratum $O_{(w^1+w^2-r,r)}$). Therefore the variety $\mathfrak{T}_2(v, w^1, r^1, w^2, r^2)$ is of pure dimension,

$$\text{(2.4.c)} \quad \dim \mathfrak{T}_2(v, w^1, r^1, w^2, r^2) \\ = w^1 w^2 + v(w^1 + w^2 - v) + \frac{1}{2}(\dim O_{(w^1-r^1,r^1)} + \dim O_{(w^2-r^2,r^2)}) \ ,$$

and the closures of $\delta_2^{-1}(O_{(w^1+w^2-r,r)})$ are irreducible components of the variety $\mathfrak{T}_2(v,w^1,r^1,w^2,r^2)$. In this way one obtains bijections

$$\begin{aligned}&\mathcal{T}_2(v,w^1,r^1,w^2,r^2)\\ &\qquad\leftrightarrow \bigsqcup_{r\in\mathbb{Z}_{\geq 0}} \mathcal{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))\times\mathcal{M}_2(v,w^1+w^2,r)\ ,\end{aligned}$$

$$\begin{aligned}&\mathcal{T}_2(w^1,r^1,w^2,r^2)\\ &\qquad\leftrightarrow \bigsqcup_{r\in\mathbb{Z}_{\geq 0}} \mathcal{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))\times\mathcal{M}_2(w^1+w^2,r)\ ,\end{aligned} \tag{2.4.d}$$

where $\mathcal{T}_2(v,w^1,r^1,w^2,r^2)$ (resp. $\mathcal{T}_2(w^1,r^1,w^2,r^2)$) is the set of irreducible components of $\mathfrak{T}_2(v,w^1,r^1,w^2,r^2)$ (resp. $\mathfrak{T}_2(w^1,r^1,w^2,r^2)$). These bijections represent the first way of labeling the set $\mathcal{T}_2(w^1,r^1,w^2,r^2)$. Another labeling of the same set is described in the next subsection.

2.5. **The tensor product diagram.** Consider the following commutative diagram:

(2.5.a)

$$\begin{tikzcd}
\mathfrak{T}_2(v,w^1,r^1,w^2,r^2) \arrow[d, "\eta_2"] \arrow[dd, bend right, "\delta_2"'] & \mathfrak{T}'_2(v,w^1,r^1,w^2,r^2) \arrow[l, "a_2"'] \arrow[r, dotted, "b_2"] \arrow[d, "\eta'_2"] & \mathfrak{M}'_2(w^1,r^1)\times\mathfrak{M}'_2(w^2,r^2) \arrow[d, "\pi_2\times\pi_2"] \\
\mathfrak{R}_2(w^1,r^1,w^2,r^2) \arrow[d, "\varkappa_2"] & \mathfrak{R}'_2(w^1,r^1,w^2,r^2) \arrow[l, "c_2"'] \arrow[r, "d_2"] & O_{(w^1-r^1,r^1)}\times O_{(w^2-r^2,r^2)} \\
\mathfrak{N}_2(w^1+w^2) & &
\end{tikzcd}$$

Here the notation is as follows:

$\mathfrak{R}_2(w^1,r^1,w^2,r^2)$ is a variety of pairs (t,X), where $t\in\mathfrak{N}_2(w^1+w^2)$, and X is a subspace of $\mathbb{C}^{w^1+w^2}$ such that $\dim X=w^1$, $tX\subset X$, $\operatorname{rank} t|_X=r^1$, $\operatorname{rank} t|_{(\mathbb{C}^{w^1+w^2}/X)}=r^2$;

$\mathfrak{T}'_2(v,w^1,r^1,w^2,r^2)$ (resp. $\mathfrak{R}'_2(w^1,r^1,w^2,r^2)$) is the set of triples (x,R,Q), where $x=(t,X,F)\in\mathfrak{T}_2(v,w^1,r^1,w^2,r^2)$ (resp. $x=(t,X)\in\mathfrak{R}_2(w^1,r^1,w^2,r^2)$), R is an isomorphism $X\to\mathbb{C}^{w^1}$, Q is an isomorphism $\mathbb{C}^{w^1+w^2}/X\to\mathbb{C}^{w^2}$;

$\mathfrak{M}'_2(w^i,r^i)=\pi_2^{-1}(O_{(w^i-r^i,r^i)})\subset\mathfrak{M}_2(w^i)$ for $i=1,2$;

$\eta_2((t,X,F))=(t,X)$;

$\eta'_2((x,R,Q))=(\eta_2(x),R,Q)$;

$\varkappa_2((t,X))=t$;

$a_2(((t,X,F),R,Q))=(t,X,F)$;

$c_2(((t,X),R,Q))=(t,X)$;

$b_2(((t,X,F),R,Q))=((Rt|_XR^{-1},R(F\cap X)),(Qt|_{\mathbb{C}^{w^1+w^2}/X}Q^{-1},Q(F/(F\cap X))))$;

$d_2(((t,X),R,Q))=(Rt|_XR^{-1},Qt|_{\mathbb{C}^{w^1+w^2}/X}Q^{-1})$.

Note that the map b_2 is not regular (this is why the dotted arrow is used in the diagram). However restricting a_2 and b_2 to

$$a_2^{-1}(\mathfrak{T}_2(v^1,w^1,r^1,v^2,w^2,r^2))=b_2^{-1}(\mathfrak{M}'_2(v^1,w^1,r^1)\times\mathfrak{M}'_2(v^2,w^2,r^2))$$

one obtains a commutative diagram in which all maps are regular:

(2.5.b)

$$\begin{tikzcd}
\mathfrak{T}_2(v^1,w^1,r^1,v^2,w^2,r^2) \arrow[d, "\eta_2"] \arrow[dd, bend right=60, "\delta_2"'] & \mathfrak{T}'_2(v^1,w^1,r^1,v^2,w^2,r^2) \arrow[l, "a_2"'] \arrow[r, "b_2"] \arrow[d, "\eta'_2"] & \mathfrak{M}'_2(v^1,w^1,r^1)\times \mathfrak{M}'_2(v^2,w^2,r^2) \arrow[d, "\pi_2\times\pi_2"] \\
\mathfrak{R}_2(w^1,r^1,w^2,r^2) \arrow[d, "\varkappa_2"] & \mathfrak{R}'_2(w^1,r^1,w^2,r^2) \arrow[l, "c_2"'] \arrow[r, "d_2"] & O_{(w^1-r^1,r^1)}\times O_{(w^2-r^2,r^2)} \\
\mathfrak{N}_2(w^1+w^2) & &
\end{tikzcd}$$

Here the notation is as follows:

$\mathfrak{T}_2(v^1,w^1,r^1,v^2,w^2,r^2)$ is a locally closed subset of the tensor product variety defined in subsection 2.3;

$\mathfrak{T}'_2(v^1,w^1,r^1,v^2,w^2,r^2) = a_2^{-1}(\mathfrak{T}_2(v^1,w^1,r^1,v^2,w^2,r^2))$;

$\mathfrak{M}'_2(v^i,w^i,r^i) = \mathfrak{M}'_2(w^i,r^i)\cap\mathfrak{M}_2(v^i,w^i)$ for $i=1,2$.

Let

$$\mathfrak{R}_2(w^1,r^1,w^2,r^2,r) = \varkappa_2^{-1}(O_{(w^1+w^2-r,r)}) ,$$
$$\mathfrak{R}'_2(w^1,r^1,w^2,r^2,r) = c_2^{-1}(\mathfrak{R}_2(w^1,r^1,w^2,r^2,r)) .$$

Proposition.

2.5.c. *in* (2.5.a) *and* (2.5.b) *the map* $\varkappa_2$ *restricted to* $\mathfrak{R}_2(w^1,r^1,w^2,r^2,r)$ *is a locally trivial fibration over* $O_{(w^1+w^2-r,r)}$ *with a fiber isomorphic to the Spaltenstein variety* $\mathfrak{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))$*;*

2.5.d. *in* (2.5.a) *and* (2.5.b) *the maps* a_2 *and* c_2 *are principal* $GL(w^1)\times GL(w^2)$*-bundles;*

2.5.e. *in* (2.5.b) *the map* b_2 *is a locally trivial fibration with a constant smooth connected fiber of dimension*

$$\dim GL(w^1)+\dim GL(w^2)+w^1w^2+v^1(w^2-v^2)+v^2(w^1-v^1) ;$$

2.5.f. *if* $r^1+r^2\le r\le\min(w^2-r^2+r^1,w^1-r^1+r^2)$*, then* d_2 *restricted to* $\mathfrak{R}'_2(w^1,r^1,w^2,r^2,r)$ *is a locally trivial fibration with a constant smooth connected fiber of dimension*

$$\begin{aligned}&\dim GL(w^1)+\dim GL(w^2)\\ &+w^1w^2+\frac{1}{2}(\dim O_{(w^1+w^2-r,r)}-\dim O_{(w^1-r^1,r^1)}-\dim O_{(w^2-r^2,r^2)}) ,\end{aligned}$$

otherwise $\mathfrak{R}'_2(w^1,r^1,w^2,r^2,r)$ *is empty;*

2.5.g. *if* $r=\min\{w^2-v^2+r^1,v^1+r^2\}$*, then* $\eta_2'^{-1}(\mathfrak{R}'_2(w^1,r^1,w^2,r^2,r))$ *is an open dense subset of* $\mathfrak{T}'_2(v^1,w^1,r^1,v^2,w^2,r^2)$*;*

2.5.h. $\mathfrak{T}'_2(v^1,w^1,r^1,v^2,w^2,r^2)$ *is a smooth connected quasi-projective variety,*

$$\begin{aligned}&\dim\mathfrak{T}'_2(v^1,w^1,r^1,v^2,w^2,r^2)=\dim GL(w^1)+\dim GL(w^2)\\ &+w^1w^2+v(w^1+w^2-v)+\frac{1}{2}(\dim O_{(w^1-r^1,r^1)}+\dim O_{(w^2-r^2,r^2)}) ,\end{aligned}$$

where $v=v^1+v^2$*;*

2.5.i. *if* $r^1 + r^2 \le r \le \min(w^2 - r^2 + r^1, w^1 - r^1 + r^2)$*, then* $\mathfrak{R}'_2(w^1, r^1, w^2, r^2, r)$ *is a smooth connected quasi-projective variety,*

$$\dim \mathfrak{R}'_2(w^1, r^1, w^2, r^2, r) = \dim GL(w^1) + \dim GL(w^2)$$
$$+ w^1 w^2 + \frac{1}{2}(\dim O_{(w^1+w^2-r,r)} + \dim O_{(w^1-r^1,r^1)} + \dim O_{(w^2-r^2,r^2)}) \ ,$$

otherwise $\mathfrak{R}'_2(w^1, r^1, w^2, r^2, r)$ *is empty.*

Proof. Statements 2.5.c and 2.5.d follow from definitions. For 2.5.e write b_2 as a composition of fibrations (cf. the proof of Proposition 2.4):

$$\mathfrak{T}'_2(v^1, w^1, r^1, v^2, w^2, r^2) \xrightarrow{p_1} \mathfrak{T}''_2(v^1, w^1, r^1, v^2, w^2, r^2)$$
$$\xrightarrow{p_2} \mathfrak{T}'''_2(v^1, w^1, r^1, v^2, w^2, r^2) \xrightarrow{p_3} \mathfrak{T}''''_2(v^1, w^1, r^1, v^2, w^2, r^2)$$
$$\xrightarrow{p_4} \mathfrak{M}'_2(v^1, w^1, r^1) \times \mathfrak{M}'_2(v^2, w^2, r^2) \ .$$

Here the notation is as follows:

$\mathfrak{T}''''_2(v^1, w^1, r^1, v^2, w^2, r^2)$ is a variety of tuples (X, t^1, F^1, t^2, F^2), where

$$((t^1, F^1), (t^2, F^2)) \in \mathfrak{M}'_2(v^1, w^1, r^1) \times \mathfrak{M}'_2(v^2, w^2, r^2)$$

and X is a subspace of $\mathbb{C}^{w^1+w^2}$ of dimension w^1;

$\mathfrak{T}'''_2(v^1, w^1, r^1, v^2, w^2, r^2)$ is a variety of tuples $(R, Q, X, t^1, F^1, t^2, F^2)$, where

$$(X, t^1, F^1, t^2, F^2) \in \mathfrak{T}''''_2(v^1, w^1, r^1, v^2, w^2, r^2)$$

and R (resp. Q) is an isomorphism $X \to \mathbb{C}^{w^1}$ (resp . $\mathbb{C}^{w^1+w^2}/X \to \mathbb{C}^{w^2}$);

$\mathfrak{T}''_2(v^1, w^1, r^1, v^2, w^2, r^2)$ is a variety of tuples $(F, R, Q, X, t^1, F^1, t^2, F^2)$, where

$$(R, Q, X, t^1, F^1, t^2, F^2) \in \mathfrak{T}'''_2(v^1, w^1, r^1, v^2, w^2, r^2)$$

and F is a subspace of $\mathbb{C}^{w^1+w^2}$ such that $R(F \cap X) = F^1$, $Q(F/(F \cap X)) = F^2$;

$$p_1(((t, X, F), R, Q))$$
$$= (F, R, Q, X, Rt|_X R^{-1}, R(F \cap X), Qt|_{(\mathbb{C}^{w^1+w^2}/X)} Q^{-1}, Q(F/(F \cap X))) \ ;$$

p_2, p_3, p_4 are natural projections.

The fiber of p_4 is a Grassmannian of dimension $w^1 w^2$, the fiber of p_3 is isomorphic to $GL(w^1) \times GL(w^2)$, the fiber of p_2 over $(R, Q, X, t^1, F^1, t^2, F^2)$ is isomorphic to the affine space $\mathrm{Hom}(Q^{-1}(F^2), X/(R^{-1}(F^1)))$, and one can describe the fiber of p_1 over $(F, R, Q, X, t^1, F^1, t^2, F^2)$ as follows. Choose a subspace $(F \cap X)^X \subset X$ (resp. $(F \cap X)^F \subset F$, $(F + X)^{\mathbb{C}^{w^1+w^2}} \subset \mathbb{C}^{w^1+w^2}$) complementary to $F \cap X$ in X (resp. to $F \cap X$ in F, to $(F + X)$ in $\mathbb{C}^{w^1+w^2}$), and consider R (resp. Q) as a map $(F \cap X)^X \oplus (F \cap X) \to \mathbb{C}^{w^1}$ (resp. $(F + X)^{\mathbb{C}^{w^1+w^2}} \oplus (F \cap X)^F \to \mathbb{C}^{w^2}$). Then

$$R^{-1} t^1 R = \begin{pmatrix} 0 & 0 \\ u^1 & 0 \end{pmatrix} \ , \qquad Q^{-1} t^2 Q = \begin{pmatrix} 0 & 0 \\ u^2 & 0 \end{pmatrix} \ ,$$

for some matrices u^1, u^2, and the fiber of p_1 consists of operators $t \in \mathrm{End}(\mathbb{C}^{w^1+w^2})$ that have the following block form

$$t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ u^2 & 0 & 0 & 0 \\ s & u^1 & 0 & 0 \end{pmatrix} \qquad \text{(2.5.j)}$$

with respect to the presentation of $\mathbb{C}^{w^1+w^2}$ as a direct sum

$$\mathbb{C}^{w^1+w^2} = (F+X)^{\mathbb{C}^{w^1+w^2}} \oplus (F\cap X)^X \oplus (F\cap X)^F \oplus (F\cap X)\ .$$

In (2.5.j) s is an arbitrary $v^1\times(w^2-v^2)$ matrix. The statement 2.5.e follows. Proof of 2.5.f is analogous. The crucial step is to describe the set of matrices of the form

$$t = \begin{pmatrix} Q^{-1}t^2Q & 0 \\ * & R^{-1}t^1R \end{pmatrix}\ ,$$

such that $t\in O_{(w^1+w^2-r,r)}$.

2.5.g follows from the fact that matrices of rank equal to $\min(w^2-v^2+r^1, v^1+r^2)$ form an open dense subset in the set of matrices of the form (2.5.j).

2.5.h follows from 2.5.e and (2.2).

2.5.i follows from 2.5.f or 2.5.c together with 2.5.d. □

Statement 2.5.d together with (2.4.c) implies that the variety $\mathfrak{T}'_2(v,w^1,r^1,w^2,r^2)$ is of pure dimension,

$$\begin{aligned}\dim\mathfrak{T}'_2(v,w^1,r^1,w^2,r^2) &= \dim GL(w^1)+\dim GL(w^2)\\ &\quad + w^1w^2 + v(w^1+w^2-v) + \frac{1}{2}(\dim O_{(w^1-r^1,r^1)} + \dim O_{(w^2-r^2,r^2)})\ ,\end{aligned}$$

and the set of irreducible components of $\mathfrak{T}'_2(v,w^1,r^1,w^2,r^2)$ is in a natural bijection with $\mathcal{T}_2(v,w^1,r^1,w^2,r^2)$ (the set of irreducible components of $\mathfrak{T}_2(v,w^1,r^1,w^2,r^2)$.

The fiber of η'_2 (in the diagram (2.5.a)) over a point in $\mathfrak{R}'_2(w^1,r^1,w^2,r^2,r)$ is isomorphic to $\mathfrak{M}_2(v,w^1+w^2,r)$. Since

$$\dim\mathfrak{R}'_2(w^1,r^1,w^2,r^2,r)+\dim\mathfrak{M}_2(v,w^1+w^2,r) = \dim\mathfrak{T}'_2(v,w^1,r^1,w^2,r^2)$$

one obtains another description of the bijection (2.4.d)

$$\begin{aligned}&\mathcal{T}_2(v,w^1,r^1,w^2,r^2)\\ &\qquad\leftrightarrow \bigsqcup_{r\in\mathbb{Z}_{\geq 0}} \mathcal{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))\times\mathcal{M}_2(v,w^1+w^2,r)\ ,\end{aligned}$$

where now $\mathcal{S}_2(((w^1,r^1),(w^2,r^2)),(w^1+w^2,r))$ represents the set of irreducible components of $\mathfrak{R}'_2(w^1,r^1,w^2,r^2,r)$ (cf. 2.5.i) or the set of irreducible components of a fiber of d_2 in the diagram (2.5.a) (cf. 2.5.f). This description of the bijection (2.4.d) as coming from the double fibration $d_2\circ\eta'_2$ is convenient for study of $gl(2)$-restriction in the gl_N case.

On the other hand $\mathfrak{T}'_2(v,w^1,r^1,w^2,r^2)$ is the total space of the double fibration $(\pi_2\times\pi_2)\circ b_2$. To make sense of b_2 one should consider the diagram (2.5.b) instead of (2.5.a). Then 2.5.h says that the variety $\mathfrak{T}'_2(v^1,w^1,r^1,v^2,w^2,r^2)$ is smooth, connected, and has the same dimension as $\mathfrak{T}'_2(v,w^1,r^1,w^2,r^2)$. Hence the closure of $\mathfrak{T}'_2(v^1,w^1,r^1,v^2,w^2,r^2)$ is an irreducible component of $\mathfrak{T}'_2(v,w^1,r^1,w^2,r^2)$, and thus one obtains the following bijections

$$\begin{aligned}\mathcal{T}_2(v^1,w^1,r^1,v^2,w^2,r^2) &\leftrightarrow \mathcal{M}_2(v^1,w^1,r^1)\times\mathcal{M}_2(v^2,w^2,r^2)\ ,\\ \mathcal{T}_2(v,w^1,r^1,w^2,r^2) &\leftrightarrow \bigsqcup_{\substack{v^1,v^2\in\mathbb{Z}_{\geq 0}\\ v^1+v^2=v}} \mathcal{M}_2(v^1,w^1,r^1)\times\mathcal{M}_2(v^2,w^2,r^2)\ ,\\ \mathcal{T}_2(w^1,r^1,w^2,r^2) &\leftrightarrow \mathcal{M}_2(w^1,r^1)\times\mathcal{M}_2(w^2,r^2)\ .\end{aligned} \tag{2.5.k}$$

Combining (2.5.k) and (2.4.d) (which represent two ways of labeling the set of irreducible components of the tensor product variety $\mathfrak{T}_2(w^1, r^1, w^2, r^2)$) one obtains a bijection

$$\begin{aligned}\tau_2 : \mathcal{M}_2(w^1, r^1) \times \mathcal{M}_2(w^2, r^2) \\ \xrightarrow{\sim} \bigsqcup_{r\in\mathbb{Z}_{\geq 0}} \mathcal{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r)) \times \mathcal{M}_2(w^1 + w^2, r) \ .\end{aligned}$$

It follows from the definition of τ_2 and 2.5.g that

$$\begin{aligned}(2.5.\mathrm{l}) \quad & \tau_2\big((\mathfrak{M}_2(v^1, w^1, r^1) \ , \ \mathfrak{M}_2(v^2, w^2, r^2))\big) \\ & = \big(\mathfrak{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r_0)) \ , \ \mathfrak{M}_2(v^1 + v^2, w^1 + w^2, r_0)\big) \ ,\end{aligned}$$

where $r_0 = \min\{w^2 - v^2 + r^1, v^1 + r^2\}$.

2.6. **The main theorem for gl_2.** The equality (2.5.l) together with definitions (1.2.a) and (2.2.d) imply the following theorem, which is the main theorem of this paper in the gl_2 case.

Theorem. *The map τ_2 (described in sections 2.4–2.5 using two labelings of the set of irreducible components of the tensor product variety $\mathfrak{T}_2(w^1, r^1, w^2, r^2)$) is a crystal isomorphism*

$$\begin{aligned}\tau_2 : \mathcal{M}_2(w^1, r^1) \otimes \mathcal{M}_2(w^2, r^2) \\ \xrightarrow{\sim} \bigoplus_{r\in\mathbb{Z}_{\geq 0}} \mathcal{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r)) \otimes \mathcal{M}_2(w^1 + w^2, r) \ ,\end{aligned}$$

where $\mathcal{S}_2(((w^1, r^1), (w^2, r^2)), (w^1 + w^2, r))$ is considered as a set with the trivial gl_2-crystal structure.

3. Spaltenstein varieties and gl_N-crystals

This section contains a geometric description of a closed family of gl_N-crystals. Throughout the section N is a fixed positive integer.

3.1. **Notation.** Let $\mathcal{Q}_m$ be the set of m-tuples of non-negative integers. Given $\mathbf{a} \in \mathcal{Q}_m$ let $\mathbf{a}_i$ denote the i-th component of $\mathbf{a}$, and $|\mathbf{a}|$ denote the sum of all components of $\mathbf{a}$: $|\mathbf{a}| = \sum_{i=1}^m \mathbf{a}_i$. Let $\mathcal{Q}_m(k) = \{\mathbf{a} \in \mathcal{Q}_m \mid |\mathbf{a}| = k\}$.

In this paper representations of gl_N are assumed integrable (polynomial). Therefore all weights are positive and one can identify the relevant part of the weight lattice of gl_N with $\mathcal{Q}_N$.

Let $\mathcal{Q}_m^+$ (resp. $\mathcal{Q}_m^+(k)$) be the set of $\mathbf{a} \in \mathcal{Q}_m$ (resp. $\mathbf{a} \in \mathcal{Q}_m(k)$) such that $\mathbf{a}_1 \geq \mathbf{a}_2 \geq \cdots \geq \mathbf{a}_m$. One can think of $\mathcal{Q}_m^+$ as of the set of partitions, or Young diagrams, or highest weights of integrable representations of gl_m. Let $L(\lambda)$ be the irreducible highest weight representation with the highest weight $\lambda \in \mathcal{Q}_N^+$, s_λ be its character (a Schur function), and $\mathcal{L}(\lambda)$ be the crystal of its canonical basis (a highest weight gl_N-crystal).

3.2. **Nilpotent orbits and Spaltenstein varieties.** Let $m \in \mathbb{Z}_{>0}$, $w \in \mathbb{Z}_{\geq 0}$, $t \in \mathrm{End}(\mathbb{C}^w)$, $t^m = 0$. Denote by $J(t) \in \mathcal{Q}_m^+(w)$ a partition given as follows: $J(t)_i$ is equal to the number of Jordan blocks of t with size greater than or equal to i ($J(t)$ is called below the Jordan type of t). For $\lambda \in \mathcal{Q}_m^+$ let

$$O_\lambda = \{t \in \mathrm{End}(\mathbb{C}^{|\lambda|}) \mid t^m = 0,\ J(t) = \lambda\} \ .$$

The set $O_\lambda \subset \mathrm{End}(\mathbb{C}^{|\lambda|})$ forms a single $GL(|\lambda|)$-orbit.

The stabilizer in $GL(|\lambda|)$ of any $t \in O_\lambda$ is connected because it is given by the compliment of a hypersurface $\det x = 0$ in the affine space $\{x \in \mathrm{End}(\mathbb{C}^{|\lambda|}) \mid xt = tx\}$. Hence O_λ is simply connected. It follows that given a locally trivial fibration over O_λ with a constant fiber of pure dimension there is a canonical bijection between the sets of irreducible components of the fiber and of the total space of the fibration. In what follows these two sets are usually denoted by the same symbol.

Let $m \in \mathbb{Z}_{>0}$, $\mathbf{w} \in \mathcal{Q}_m$. Denote by $\mathfrak{F}_m(\mathbf{w})$ the partial flag variety

$$\mathfrak{F}_m(\mathbf{w}) = \{\mathbf{F} = (\{0\} = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \ldots \subset \mathbf{F}_m = \mathbb{C}^{|\mathbf{w}|}) \mid \dim(\mathbf{F}_i/\mathbf{F}_{i-1}) = \mathbf{w}_i\}$$

and let $\mathfrak{F}_m(w) = \bigsqcup_{\mathbf{w} \in \mathcal{Q}_m(w)} \mathfrak{F}_m(\mathbf{w})$ be the variety of all m-step partial flags in $\mathbb{C}^w$. Given $\mathbf{F} \in \mathfrak{F}_m(w)$ let $\dim \mathbf{F}$ denote the m-tuple of integers consisting of dimensions of subfactors of the flag $\mathbf{F}$. Thus $\mathbf{F} \in \mathfrak{F}_m(\mathbf{w})$ if and only if $\dim \mathbf{F} = \mathbf{w}$.

Let $l, m \in \mathbb{Z}_{>0}$, $\mu \in \mathcal{Q}_m^+$, Λ be an l-tuple $(\lambda^1, \ldots, \lambda^l)$, where $\lambda^i \in \mathcal{Q}_m^+$ and $\sum_{i=1}^l |\lambda^i| = |\mu|$. Let $|\Lambda|$ denote an element of $\mathcal{Q}_l$ given by $|\Lambda| = (|\lambda^1|, \ldots, |\lambda^l|)$. Choose $t \in O_\mu$ and consider the following subvariety of $\mathfrak{F}_l(|\Lambda|)$:

$$\mathfrak{S}_l(\Lambda, \mu) = \{\mathbf{F} \in \mathfrak{F}_l(|\Lambda|) \mid t\mathbf{F}_i \subset \mathbf{F}_i,\ t|_{\mathbf{F}_i/\mathbf{F}_{i-1}} \in O_{\lambda^i}\} \ .$$

The variety $\mathfrak{S}_l(\Lambda, \mu)$ does not depend (up to an isomorphism) on the choice of $t \in O_\mu$. However when the choice of t is important the notation is $\mathfrak{S}_l(\Lambda, t)$ instead of $\mathfrak{S}_l(\Lambda, \mu)$.

Theorem. [Spa82, II.5] *The variety $\mathfrak{S}_l(\Lambda, \mu)$ is of pure dimension,*

$$\begin{aligned} 2 \dim \mathfrak{S}_l(\Lambda, \mu) &= |\mu|(|\mu| - 1) - \sum_{i=1}^l |\lambda^i|(|\lambda^i| - 1) - \dim O_\mu + \sum_{i=1}^l \dim O_{\lambda^i} \\ &= \sum_{\substack{i,j=1 \\ i \neq j}}^l |\lambda^i||\lambda^j| - \dim O_\mu + \sum_{i=1}^l \dim O_{\lambda^i} \ . \end{aligned}$$

The variety $\mathfrak{S}_l(\Lambda, \mu)$ is called (l-step) Spaltenstein variety. Let $\mathcal{S}_l(\Lambda, \mu)$ denote the set of irreducible components of $\mathfrak{S}_l(\Lambda, \mu)$.

3.3. **The Hall theorem.** Let $\mu^1, \mu^2 \in \mathcal{Q}_N^+$. One has the following (tensor product) decompositions in the ring of symmetric functions, the category of integrable representations of gl_N, and the category of gl_N-crystals:

$$\begin{aligned} s_{\mu^1} s_{\mu^2} &= \sum_{\lambda \in \mathcal{Q}_N^+(|\mu^1| + |\mu^2|)} c^\lambda_{\mu^1 \mu^2} s_\lambda \ , \\ L(\mu^1) \otimes L(\mu^2) &= \bigoplus_{\lambda \in \mathcal{Q}_N^+(|\mu^1| + |\mu^2|)} C((\mu^1, \mu^2), \lambda) \otimes L(\lambda) \ , \\ \mathcal{L}(\mu^1) \otimes \mathcal{L}(\mu^2) &= \bigoplus_{\lambda \in \mathcal{Q}_N^+(|\mu^1| + |\mu^2|)} \mathcal{C}((\mu^1, \mu^2), \lambda) \otimes \mathcal{L}(\lambda) \ , \end{aligned}$$

where $c^{\lambda}_{\mu^1\mu^2}$ is a non-negative integer, called the Littlewood-Richardson coefficient, $C((\mu^1,\mu^2),\lambda)$ is a linear space of dimension $c^{\lambda}_{\mu^1\mu^2}$ with the trivial gl_N-action, and $\mathcal{C}((\mu^1,\mu^2),\lambda)$ is a trivial gl_N-crystal (i.e. a set with the trivial crystal structure) of cardinal $c^{\lambda}_{\mu^1\mu^2}$.

The following theorem is due to Hall [Hal59].

Theorem. *The number of irreducible components of the (2-step) Spaltenstein variety $\mathfrak{S}_2((\mu^1,\mu^2),\lambda)$ is equal to the Littlewood-Richardson coefficient $c^{\lambda}_{\mu^1\mu^2}$. In other words, the set $\mathcal{S}_2((\mu^1,\mu^2),\lambda)$ is isomorphic to the set $\mathcal{C}((\mu^1,\mu^2),\lambda)$ as a trivial gl_N-crystal.*

Remark. Let $\mathbb{F}_q$ be a finite field with q elements. The original Hall's statement [Hal59] says that the number of $\mathbb{F}_q$-rational points of the variety $\mathfrak{S}_2((\mu^1,\mu^2),\lambda)$ is given by a polynomial in q with the leading coefficient equal to $c^{\lambda}_{\mu^1\mu^2}$ (the variety $\mathfrak{S}_2((\mu^1,\mu^2),\lambda)$ is defined over any field). The above formulation of the theorem follows from this one by a weak form of the Weil conjectures (cf. [LW54]).

3.4. **A special case.** The following lemma is a special case of the Hall theorem.

Lemma. *If $\lambda=\mu^1+\mu^2$ (i.e. $\lambda_i=\mu^1_i+\mu^2_i$) then the variety $\mathfrak{S}_2((\mu^1,\mu^2),\lambda)$ is non-empty.*

Proof. Let $t\in O_{\mu^1+\mu^2}$. Choose a basis $\{e^j_i\}^{1\le j\le \mu^1_i+\mu^2_i}_{1\le i\le N}$ in $\mathbb{C}^{|\mu^1|+|\mu^2|}$ such that

$$te^j_i=\begin{cases} e^j_{i-1} & \text{if } i>1\ ,\\ 0 & \text{if } i=1\ ,\end{cases}$$

and consider a subspace $X\subset\mathbb{C}^{|\mu^1|+|\mu^2|}$ defined as the linear span of the set

$$\begin{aligned}\{e^j_i\quad|\quad & 1\le i\le N\ ,\\ & j\in\mathbb{Z}\cap\big((0,\mu^1_N]\cup(\mu^1_N+\mu^2_N,\mu^1_{N-1}+\mu^2_N]\cup\ldots\cup(\mu^1_{i+1}+\mu^2_{i+1},\mu^1_i+\mu^2_{i+1}]\big)\}\ ,\end{aligned}$$

where it is assumed that $\mu^1_{N+1}=\mu^2_{N+1}=0$. Then the flag $(\{0\}\subset X\subset\mathbb{C}^{|\mu^1|+|\mu^2|})$ belongs to $\mathfrak{S}_2((\mu^1,\mu^2),\mu^1+\mu^2)$. $\square$

One of the purposes of this paper is to explain appearance of the varieties $\mathfrak{S}_2((\mu^1,\mu^2),\lambda)$ in the tensor product decomposition for gl_N. As an application the Hall theorem is deduced in 3.11 from Lemma 3.4 and Theorem 1.3. The starting point is a geometric construction of the highest weight crystal $\mathcal{L}(\lambda)$ due to Ginzburg.

3.5. **Ginzburg's construction.** [Gin91] Given $w\in\mathbb{Z}_{\ge0}$ and $\mathbf{v}\in\mathcal{Q}_N(w)$ consider the following varieties

$\mathfrak{N}_N(w)=\{t\in\mathrm{End}(\mathbb{C}^w)\mid t^N=0\}$,

$\mathfrak{M}_N(\mathbf{v})=\{(t,\mathbf{F})\mid t\in\mathfrak{N}_N(w),\ \mathbf{F}\in\mathfrak{S}_N(\mathbf{0},t)\}$, where $\mathbf{0}=(0,\ldots,0)$, $|\mathbf{0}|=\mathbf{v}$ (in other words, $\mathfrak{M}_N(\mathbf{v})$ is the variety of pairs $(t,\mathbf{F})$, where t is a nilpotent operator and $\mathbf{F}$ is an N-step partial flag of dimension $\mathbf{v}$, such that $t\mathbf{F}_i\subset\mathbf{F}_{i-1}$),

$\mathfrak{M}_N(w) = \bigsqcup_{\mathbf{v}\in\mathcal{Q}_N(w)} \mathfrak{M}_N(\mathbf{v})$. Consider a natural map

$$\begin{array}{c}\mathfrak{M}_N(w)\\ \downarrow{\scriptstyle \pi_N}\\ \mathfrak{N}_N(w)\ ,\end{array}$$

given by $\pi_N((t,\mathbf{F})) = t$, and put $\mathfrak{M}_N(w,t) = \pi_N^{-1}(t)$. One has

$$\mathfrak{M}_N(w,t) = \bigsqcup_{\mathbf{v}\in\mathcal{Q}_N(w)} \mathfrak{M}_N(\mathbf{v},t)\ ,$$

where

$$\mathfrak{M}_N(\mathbf{v},t) = \mathfrak{M}_N(w,t) \cap \mathfrak{M}_N(\mathbf{v})\ .$$

The variety $\mathfrak{M}_N(\mathbf{v},t)$ is isomorphic to the Spaltenstein variety $\mathfrak{S}_N(\mathbf{0},t)$. It depends (up to an isomorphism) only on $\lambda = J(t)$ (the Jordan form of t, cf. 3.2). Thus the notation $\mathfrak{M}_N(\mathbf{v},\lambda)$ is often used instead of $\mathfrak{M}_N(\mathbf{v},t)$. According to 3.2 the variety $\mathfrak{M}_N(\mathbf{v},\lambda)$ is of pure dimension and

$$\dim \mathfrak{M}_N(\mathbf{v},\lambda) = \frac{1}{2}\Big(\sum_{i\neq j} \mathbf{v}_i\mathbf{v}_j - \dim O_\lambda\Big)\ . \tag{3.5.a}$$

Let $\mathcal{M}_N(\mathbf{v},\lambda)$ be the set of irreducible components of $\mathfrak{M}_N(\mathbf{v},\lambda)$ and $\mathcal{M}_N(\lambda) = \bigsqcup_{\mathbf{v}\in\mathcal{Q}_N(|\lambda|)} \mathcal{M}_N(\mathbf{v},\lambda)$ be the set of irreducible components of $\mathfrak{M}_N(|\lambda|,\lambda)$.

The set $\mathcal{M}_N(\lambda)$ can be endowed with a structure of a normal gl_N-crystal. The weight function is given by

$$wt(Z) = \mathbf{v} \quad \text{if } Z \in \mathcal{M}_N(\mathbf{v},\lambda)\ . \tag{3.5.b}$$

The functions ε_k and φ_k, and the action of the Kashiwara operators $\tilde{e}_k$ and $\tilde{f}_k$ on $\mathcal{M}_N(\lambda)$ are defined in the next two subsections using gl_2-restriction.

3.6. **Restriction to gl_2.** Throughout this subsection k denotes a fixed integer such that $1 \le k \le N-1$.

Let $\mathcal{Q}_N^k$ be the set of $(N-1)$-tuples of non-negative integers labeled as follows: $(\mathbf{u}_1,\dots,\mathbf{u}_{k-1},\mathbf{u}_{k+1},\dots,\mathbf{u}_N)$. There is a natural map $\rho_N^k : \mathcal{Q}_N \to \mathcal{Q}_N^k$ given by

$$\rho_N^k(\mathbf{v}) = (\mathbf{v}_1,\dots,\mathbf{v}_{k-1},\mathbf{v}_k+\mathbf{v}_{k+1},\dots,\mathbf{v}_N)\ .$$

Let $\mathcal{Q}_N^k(w) = \rho_N^k(\mathcal{Q}_N(w)) = \{\mathbf{u}\in\mathcal{Q}_N^k \mid \sum \mathbf{u}_i = w\}$.

Given $\mathbf{u}\in\mathcal{Q}_N^k$ consider the following partial flag variety

$$\begin{aligned}\mathfrak{F}_N^k(\mathbf{u}) = \{&\mathbf{F} = (\{0\} = \mathbf{F}_0 \subset \mathbf{F}_1 \subset \ldots \subset \mathbf{F}_{k-1} \subset \mathbf{F}_{k+1} \subset \ldots \subset \mathbf{F}_N = \mathbb{C}^{|\mathbf{u}|}) \mid \\ &\dim(\mathbf{F}_i/\mathbf{F}_{i-1}) = \mathbf{u}_i \text{ for } i\neq 0,k,k+1\ ,\ \dim(\mathbf{F}_{k+1}/\mathbf{F}_{k-1}) = \mathbf{u}_{k+1}\}\ ,\end{aligned}$$

and, similarly, the following variants of the varieties $\mathfrak{M}_N(\mathbf{v})$ and $\mathfrak{M}_N(w)$:

$$\begin{aligned}\mathfrak{M}_N^k(\mathbf{u}) = \{&(t,\mathbf{F}) \mid t\in\mathfrak{N}_N(|\mathbf{u}|),\ \mathbf{F}\in\mathfrak{F}_N^k(\mathbf{u})\ ,\\ &t\mathbf{F}_i \subset \mathbf{F}_{i-1} \text{ for } i\neq 0,k,k+1\ ,\ t\mathbf{F}_{k+1}\subset\mathbf{F}_{k+1}\ ,\ t^2\mathbf{F}_{k+1}\subset\mathbf{F}_{k-1}\}\ ,\end{aligned}$$

$$\mathfrak{M}_N^k(w) = \bigsqcup_{\mathbf{u}\in\mathcal{Q}_N^k(w)} \mathfrak{M}_N^k(\mathbf{u})\ .$$

One has the following commutative diagram:

(3.6.a)
$$\begin{tikzcd} & \mathfrak{M}_N(\mathbf{v}) \arrow[dl, "\sigma_N^k"'] \arrow[dd, "\pi_N"] \\ \mathfrak{M}_N^k(\rho_N^k(\mathbf{v})) \arrow[dr, "\pi_N^k"'] & \\ & \mathfrak{N}_N(|\mathbf{v}|) \end{tikzcd}$$

Here σ_N^k forgets the k-th subspace of the flag, and π_N^k forgets the rest of the flag.

Given $\mathbf{u} \in \mathcal{Q}_N^k$, $\mu \in \mathcal{Q}_N^+(|\mathbf{u}|)$, and $r \in \mathbb{Z}_{\geq 0}$, choose $t \in O_\mu$ and let

$$\begin{aligned} \mathfrak{M}_N^k(\mathbf{u},\mu) &= (\pi_N^k)^{-1}(t) \ , \\ \mathfrak{M}_N^k(\mathbf{u},\mu,r) &= \{(t,\mathbf{F}) \in \mathfrak{M}_N^k(\mathbf{u},\mu) \mid \operatorname{rank}(t|_{\mathbf{F}_{k+1}/\mathbf{F}_{k-1}}) = r\} \ . \end{aligned}$$

The varieties $\mathfrak{M}_N^k(\mathbf{u},\mu)$ and $\mathfrak{M}_N^k(\mathbf{u},\mu,r)$ do not depend (up to an isomorphism) on $t \in O_\mu$. The latter one is an example of the $(N-1)$-step Spaltenstein variety. More exactly, $\mathfrak{M}_N^k(\mathbf{u},\mu,r) = \mathfrak{S}_{N-1}(\Lambda,\mu)$, where $\Lambda = (0,\ldots,0,\lambda^k,0,\ldots,0)$ with $\lambda^k = (\mathbf{u}_{k+1}-r,r)$. Hence $\mathfrak{M}_N^k(\mathbf{u},\mu,r)$ is of pure dimension and

$$\dim \mathfrak{M}_N^k(\mathbf{u},\mu,r) = \frac{1}{2}\Big(\sum_{\substack{i,j=1\\ i,j\neq k\\ i\neq j}}^{N} \mathbf{u}_i\mathbf{u}_j - \dim O_\mu + \dim O_{(\mathbf{u}_{k+1}-r,r)}\Big) \ . \tag{3.6.b}$$

Let $\mathcal{M}_N^k(\mathbf{u},\mu,r)$ be the set of irreducible components of $\mathfrak{M}_N^k(\mathbf{u},\mu,r)$.

One has the following stratification

$$\mathfrak{M}_N^k(\mathbf{u},\mu) = \bigcup_{r\in\mathbb{Z}_{\geq 0}} \mathfrak{M}_N^k(\mathbf{u},\mu,r)$$

with only finitely many non-empty strata. The map σ_N^k (cf. (3.6.a)) restricted to a stratum $\mathfrak{M}_N^k(\rho_N^k(\mathbf{v}),\mu,r)$ is a locally trivial fibration with the fiber isomorphic to the fiber of the map

$$\begin{tikzcd} \mathfrak{M}_2(\mathbf{v}_k+\mathbf{v}_{k+1}) \arrow[d, "\pi_2"] \\ \mathfrak{N}_2(\mathbf{v}_k+\mathbf{v}_{k+1}) \end{tikzcd}$$

over a point in $O_{(\mathbf{v}_{k+1}+\mathbf{v}_k-r,r)}$ (see 2.2 for relevant notation). In other words the fiber is a Grassmannian and

$$\dim((\sigma_N^k)^{-1}(x)) = \mathbf{v}_k\mathbf{v}_{k+1} - \frac{1}{2}(\dim O_{(\mathbf{v}_k+\mathbf{v}_{k+1}-r,r)}) \ ,$$

where $x \in \mathfrak{M}_N^k(\rho_N^k(\mathbf{v}),\mu,r)$. Hence the dimension of $\mathfrak{M}_N^k(\rho_N^k(\mathbf{v}),\mu,r)$ plus the dimension of the fiber of σ_N^k over a point in this stratum is equal to the dimension of $\mathfrak{M}_N(\mathbf{v},\mu)$ (cf. (3.5.a)) and one has a bijection of the sets of irreducible components

$$\theta_N^k : \bigsqcup_{\mathbf{v}=(\rho_N^k)^{-1}(\mathbf{u})} \mathcal{M}_N(\mathbf{v},\mu) \xrightarrow{\sim} \bigsqcup_r \mathcal{M}_N^k(\mathbf{u},\mu,r) \times \mathcal{M}_2(\mathbf{u}_{k+1},r) \ ,$$

or

$$\theta^k_{N} : \mathcal{M}_{N}(\mu) \xrightarrow{\sim} \bigsqcup_{\substack{\mathbf{u}\in\mathcal{Q}^k_N \\ r\in\mathbb{Z}_{\geq 0}}} \mathcal{M}^k_{N}(\mathbf{u},\mu,r) \times \mathcal{M}_2(\mathbf{u}_{k+1},r) \ . \tag{3.6.c}$$

3.7. **The crystal structure.** Now one can finish the definition of the crystal structure on the set $\mathcal{M}_{N}(\mu)$. Namely, consider the RHS of (3.6.c) as a gl_2-crystal with the crystal structure coming from the second multiple, and let

$$\begin{aligned}
\varepsilon_k &= \varepsilon \circ \theta^k_{N} \ , \\
\varphi_k &= \varphi \circ \theta^k_{N} \ , \\
\tilde{e}_k &= (\theta^k_{N})^{-1} \circ \tilde{e} \circ \theta^k_{N} \ , \\
\tilde{f}_k &= (\theta^k_{N})^{-1} \circ \tilde{f} \circ \theta^k_{N} \ .
\end{aligned}$$

These formulas together with the weight function (3.5.b) provide a structure of gl_{N}-crystal on $\mathcal{M}_{N}(\mu)$. By abuse of notation this crystal is also denoted by $\mathcal{M}_{N}(\mu)$.

Proposition. *The crystal $\mathcal{M}_{N}(\mu)$ is a highest weight normal gl_{N}-crystal with the highest weight μ.*

Proof. The fact that $\mathcal{M}_{N}(\mu)$ is a normal crystal follows immediately from definitions. To prove that it is highest weight it is enough to show that it contains a unique element Z such that

$$\tilde{e}_k Z = 0 \quad \text{for } 1 \leq k \leq N-1 \ . \tag{3.7.a}$$

Fix $t \in O_\mu$, and let $(t, \mathbf{F})$ be a generic point of a component Z of $\mathfrak{M}_{N}(t)$ satisfying (3.7.a) ("generic" here means "not lying in any other component"). By definition of the operators $\tilde{e}_k$ it implies that

$$\ker t|_{(\mathbf{F}_{k+1}/\mathbf{F}_{k-1})} = \mathbf{F}_k/\mathbf{F}_{k-1} \quad \text{for } 1 \leq k \leq N-1 \ .$$

From this it follows by induction in k that

$$\ker t^{k-1}|_{\mathbf{F}_k} = \mathbf{F}_{k-1} \quad \text{for } 2 \leq k \leq N \ ,$$

and then using inverse induction in k one obtains

$$\mathbf{F}_k = \ker t^k \quad \text{for } 1 \leq k \leq N \ .$$

Hence Z has one generic point $(t, (\{0\} \subset \ker t \subset \ldots \subset \ker t^N = \mathbb{C}^{|\mu|}))$. It means that there is unique highest weight component Z of $\mathfrak{M}_{N}(|\mu|)$, Z is the point, and $wt(Z) = \mu$. The proposition follows. ∎

Actually the crystal $\mathcal{M}_{N}(\mu)$ is isomorphic to $\mathcal{L}(\mu)$ (the crystal of the canonical basis of the irreducible highest weight module with highest weight μ). One can prove it using a slightly modified argument of [Gin91]. Another proof (based on Theorem 1.3) is given in subsection 3.11.

The above description of the crystal structure via gl_2-restriction is essentially due to Lusztig [Lus91, 12] (in the more general case of quiver varieties), and was used later by Kashiwara and Saito [KS97], and Nakajima [Nak98].

3.8. **The tensor product variety and the tensor product diagram.** Let $\mu^1, \mu^2 \in \mathcal{Q}_N^+$. Consider the following variety:

$$\begin{aligned}&\mathfrak{T}_N(\mu^1,\mu^2)\\&=\{(t,X,\mathbf{F}) \mid t\in\mathfrak{N}_N(|\mu^1|+|\mu^2|),\ (t,\mathbf{F})\in\mathfrak{M}_N(|\mu^1|+|\mu^2|),\ X\in\mathfrak{S}_2((\mu^1,\mu^2),t)\}\ .\end{aligned}$$

In plain words a point of $\mathfrak{T}_N(\mu^1,\mu^2)$ is a triple consisting of a nilpotent operator $t \in \mathrm{End}(\mathbb{C}^{|\mu^1|+|\mu^2|})$, an N-step partial flag $\mathbf{F}$ in $\mathbb{C}^{|\mu^1|+|\mu^2|}$, and a subspace X in $\mathbb{C}^{|\mu^1|+|\mu^2|}$ of dimension $|\mu^1|$, such that t preserves both $\mathbf{F}$ and X, and when restricted to X (resp. $\mathbb{C}^{|\mu^1|+|\mu^2|}/X$, subfactors of $\mathbf{F}$) gives a nilpotent operator of Jordan type μ^1 (resp. a nilpotent operator of Jordan type μ^2, 0 operator).

The variety $\mathfrak{T}_N(\mu^1,\mu^2)$ is called *the tensor product variety* (for gl_N). It plays an important role in geometric description of the tensor product. In particular using two different ways of labeling irreducible components of $\mathfrak{T}_N(\mu^1,\mu^2)$ one can describe the direct sum decomposition of the tensor product of gl_N-crystals $\mathcal{M}_N(\mu^1)$ and $\mathcal{M}_N(\mu^2)$. Details are given below.

The variety $\mathfrak{T}_N(\mu^1,\mu^2)$ is a disjoint union of subvarieties labeled by the dimensions of the subspaces of $\mathbf{F}$:

$$\mathfrak{T}_N(\mu^1,\mu^2) = \bigsqcup_{\mathbf{v}\in\mathcal{Q}_N(|\mu^1|+|\mu^2|)} \mathfrak{T}_N(\mathbf{v},\mu^1,\mu^2)\ ,$$

where

$$\mathfrak{T}_N(\mathbf{v},\mu^1,\mu^2) = \{(t,X,\mathbf{F}) \in \mathfrak{T}_N(\mu^1,\mu^2) \mid \dim\mathbf{F} = \mathbf{v}\}\ .$$

There is a natural map:

$$\begin{array}{c}\mathfrak{T}_N(\mathbf{v},\mu^1,\mu^2)\\ \Big\downarrow{\scriptstyle \delta_N}\\ \mathfrak{N}_N(|\mu^1|+|\mu^2|)\ ,\end{array}$$

given by $\delta_N(t,X,\mathbf{F}) = t$. Stratify $\mathfrak{N}_N(|\mu^1|+|\mu^2|)$ by $GL(|\mu^1|+|\mu^2|)$-orbits. The map δ_N restricted to an orbit O_λ is a fibration with a constant fiber isomorphic to the direct product of two Spaltenstein varieties (for 2-step and N-step flags):

$$\delta_N^{-1}(x) \approx \mathfrak{S}_2((\mu^1,\mu^2),\lambda)\times\mathfrak{M}_N(\mathbf{v},\lambda)$$

if $x \in O_\lambda$. Therefore

$$\dim(\delta_N^{-1}(O_\lambda)) = |\mu^1||\mu^2| + \frac{1}{2}(\sum_{i\neq j}\mathbf{v}_i\mathbf{v}_j + \dim O_{\mu^1} + \dim O_{\mu^2})\ .$$

Since the dimension of $\delta_N^{-1}(O_\lambda)$ does not depend on λ, it follows that the variety $\mathfrak{T}_N(\mathbf{v},\mu^1,\mu^2)$ is of pure dimension,

$$\text{(3.8.a)}\qquad \dim\mathfrak{T}_N(\mathbf{v},\mu^1,\mu^2) = |\mu^1||\mu^2| + \frac{1}{2}(\sum_{i\neq j}\mathbf{v}_i\mathbf{v}_j + \dim O_{\mu^1} + \dim O_{\mu^2})\ ,$$

and one has the following bijection

$$\text{(3.8.b)}\qquad \mathcal{T}_N(\mathbf{v},\mu^1,\mu^2) \leftrightarrow \bigsqcup_{\lambda\in\mathcal{Q}_N^+(|\mu^1|+|\mu^2|)} \mathcal{S}_2((\mu^1,\mu^2),\lambda)\times\mathcal{M}_N(\mathbf{v},\lambda)\ ,$$

where $\mathcal{T}_N(\mathbf{v},\mu^1,\mu^2)$ is the set of irreducible components of $\mathfrak{T}_N(\mathbf{v},\mu^1,\mu^2)$.

Another way to label irreducible components of $\mathfrak{T}_N(\mu^1,\mu^2)$ is as follows. Let $\mathbf{F}\cap X$ (resp. $\mathbf{F}/(\mathbf{F}\cap X)$) denote the flag in X (resp. in $\mathbb{C}^{|\mu^1|+|\mu^2|}/X$) obtained by taking intersections with X (resp. quotients by intersections with X) of the subspaces of $\mathbf{F}$, and let

$$\begin{aligned}&\mathfrak{T}_N(\mathbf{v}^1,\mathbf{v}^2,\mu^1,\mu^2)\\ &\qquad=\{(t,X,\mathbf{F})\in\mathfrak{T}_N(\mu^1,\mu^2)\mid \dim(\mathbf{F}\cap X)=\mathbf{v}^1,\ \dim(\mathbf{F}/(\mathbf{F}\cap X))=\mathbf{v}^2\}\ .\end{aligned}$$

Each subset $\mathfrak{T}_N(\mathbf{v}^1,\mathbf{v}^2,\mu^1,\mu^2)\subset\mathfrak{T}_N(\mu^1,\mu^2)$ is locally closed and one has

$$\mathfrak{T}_N(\mathbf{v},\mu^1,\mu^2)=\bigsqcup_{\substack{\mathbf{v}^1\in\mathcal{Q}_N(|\mu^1|)\\ \mathbf{v}^2\in\mathcal{Q}_N(|\mu^2|)\\ \mathbf{v}^1+\mathbf{v}^2=\mathbf{v}}}\mathfrak{T}_N(\mathbf{v}^1,\mathbf{v}^2,\mu^1,\mu^2)\ .$$

Consider the following commutative diagram (cf. (2.5.a)):

(3.8.c)
$$\begin{tikzcd}
\mathfrak{T}_N(\mathbf{v},\mu^1,\mu^2) \arrow[d, "\eta_N"] \arrow[dd, bend right=60, "\delta_N"'] & \mathfrak{T}'_N(\mathbf{v},\mu^1,\mu^2) \arrow[l, "a_N"'] \arrow[r, dotted, "b_N"] \arrow[d, "\eta'_N"] & \mathfrak{M}'_N(\mu^1)\times\mathfrak{M}'_N(\mu^2) \arrow[d, "\pi_N\times\pi_N"] \\
\mathfrak{R}_N(\mu^1,\mu^2) \arrow[d, "\varkappa_N"] & \mathfrak{R}'_N(\mu^1,\mu^2) \arrow[l, "c_N"'] \arrow[r, "d_N"] & O_{\mu^1}\times O_{\mu^2} \\
\mathfrak{N}_N(|\mu^1|+|\mu^2|) & &
\end{tikzcd}$$

Here the notation is as follows:

$\mathfrak{R}_N(\mu^1,\mu^2)$ is the variety of pairs (t,X), where $t\in\mathfrak{N}_N(|\mu^1|+|\mu^2|)$, and X is a subspace of $\mathbb{C}^{|\mu^1|+|\mu^2|}$ such that $\dim X=|\mu^1|$, $tX\subset X$, $t|_X$ has Jordan type μ^1, $t|_{(\mathbb{C}^{w^1+w^2}/X)}$ has Jordan type μ^2;

$\mathfrak{T}'_N(\mathbf{v},\mu^1,\mu^2)$ (resp. $\mathfrak{R}'_N(\mu^1,\mu^2)$) is the variety of triples (x,R,Q), where $x=(t,X,\mathbf{F})\in\mathfrak{T}_N(\mathbf{v},\mu^1,\mu^2)$ (resp. $x=(t,X)\in\mathfrak{R}_N(\mu^1,\mu^2)$), R (resp. Q) is an isomorphism $X\to\mathbb{C}^{|\mu^1|}$ (resp. $\mathbb{C}^{|\mu^1|+|\mu^2|}/X\to\mathbb{C}^{|\mu^2|}$);

$\mathfrak{M}'_N(\mu^i)=\pi_N^{-1}(O_{\mu^i})\subset\mathfrak{M}_N(|\mu^i|)$ for $i=1,2$;

$\eta_N((t,X,\mathbf{F}))=(t,X)$;

$\eta'_N((x,R,Q))=(\eta_N(x),R,Q)$;

$\varkappa_N((t,X))=t$;

$a_N(((t,X,\mathbf{F}),R,Q))=(t,X,\mathbf{F})$;

$c_N(((t,X),R,Q))=(t,X)$;

$b_N(((t,X,\mathbf{F}),R,Q))=((Rt|_XR^{-1},R(\mathbf{F}\cap X)),(Qt|_{\mathbb{C}^{|\mu^1|+|\mu^2|}/X}Q^{-1},Q(\mathbf{F}/(\mathbf{F}\cap X))))$;

$d_N(((t,X),R,Q))=(Rt|_XR^{-1},Qt|_{\mathbb{C}^{w^1+w^2}/X}Q^{-1})$.

The map b_N is not regular. To fix it one has to restrict a_N and b_N to the locally closed subset

$$\begin{aligned}\mathfrak{T}'_N(\mathbf{v}^1,\mathbf{v}^2,\mu^1,\mu^2)&=a_N^{-1}(\mathfrak{T}_N(\mathbf{v}^1,\mathbf{v}^2,\mu^1,\mu^2))\\ &=b_N^{-1}(\mathfrak{M}'_N(\mathbf{v}^1,\mu^1)\times\mathfrak{M}'_N(\mathbf{v}^2,\mu^2))\ .\end{aligned}$$

The result is the following diagram (cf. (2.5.b)):

(3.8.d)

$$\begin{tikzcd}
\mathfrak{T}_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2) \arrow[d, "\eta_N"] \arrow[dd, bend right=60, "\delta_N"'] & \mathfrak{T}'_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2) \arrow[l, "a_N"'] \arrow[r, "b_N"] \arrow[d, "\eta'_N"] & \mathfrak{M}'_N(\mathbf{v}^1, \mu^1) \times \mathfrak{M}'_N(\mathbf{v}^2, \mu^2) \arrow[d, "\pi_N \times \pi_N"] \\
\mathfrak{R}_N(\mu^1, \mu^2) \arrow[d, "\varkappa_N"] & \mathfrak{R}'_N(\mu^1, \mu^2) \arrow[l, "c_N"'] \arrow[r, "d_N"] & O_{\mu^1} \times O_{\mu^2} \\
\mathfrak{N}_N(|\mu^1| + |\mu^2|) & &
\end{tikzcd}$$

The following proposition is an analogue of Proposition 2.5.

Proposition.

3.8.e. *in the diagrams* (3.8.c) *and* (3.8.d) *the maps* a_N *and* c_N *are principal* $GL(|\mu^1|) \times GL(|\mu^2|)$*-bundles;*

3.8.f. *in the diagram* (3.8.d) *the map* b_N *is a locally trivial fibration with a constant smooth connected fiber of dimension*

$$\dim(GL(|\mu^1|)) + \dim(GL(|\mu^2|)) + |\mu_1||\mu_2| + \sum_{i \neq j} \mathbf{v}_i^1 \mathbf{v}_j^2 \ .$$

Proof. Statement 3.8.e follows from definitions. To prove 3.8.f write b_N as a composition of fibrations (cf. the proof of 2.5.e):

$$\begin{aligned}
\mathfrak{T}'_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2) &\xrightarrow{q_1} \mathfrak{T}''_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2) \xrightarrow{q_2} \mathfrak{T}'''_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2) \\
&\xrightarrow{q_3} \mathfrak{T}''''_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2) \xrightarrow{q_4} \mathfrak{M}'_N(\mathbf{v}^1, \mu^1) \times \mathfrak{M}'_N(\mathbf{v}^2, \mu^2) \ .
\end{aligned}$$

Here the notation is as follows:

$\mathfrak{T}''''_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2)$ is a variety of tuples $(X, \mathbf{F}^1, \mathbf{F}^2, t^1, t^2)$, where

$$((\mathbf{F}^1, t^1), (\mathbf{F}^2, t^2)) \in \mathfrak{M}'_N(\mathbf{v}^1, \mu^1) \times \mathfrak{M}'_N(\mathbf{v}^2, \mu^2)$$

and X is a subspace of $\mathbb{C}^{|\mu^1|+|\mu^2|}$ of dimension $|\mu^1|$;

$\mathfrak{T}'''_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2)$ is a variety of tuples $(R, Q, X, \mathbf{F}^1, \mathbf{F}^2, t^1, t^2)$, where

$$(X, \mathbf{F}^1, \mathbf{F}^2, t^1, t^2) \in \mathfrak{T}''''_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2)$$

and R (resp. Q) is an isomorphism $X \to \mathbb{C}^{|\mu^1|}$ (resp. $\mathbb{C}^{|\mu^1|+|\mu^2|}/X \to \mathbb{C}^{|\mu^2|}$);

$\mathfrak{T}''_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2)$ is a variety of tuples $(\mathbf{F}, R, Q, X, \mathbf{F}^1, \mathbf{F}^2, t^1, t^2)$, where

$$(R, Q, X, \mathbf{F}^1, \mathbf{F}^2, t^1, t^2) \in \mathfrak{T}'''_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2)$$

and $\mathbf{F}$ is an N-step flag in $\mathbb{C}^{|\mu^1|+|\mu^2|}$ such that $R(\mathbf{F} \cap X) = \mathbf{F}^1$, $Q(\mathbf{F}/(\mathbf{F} \cap X)) = \mathbf{F}^2$;

$$\begin{aligned}
&q_1(((t, X, \mathbf{F}), R, Q)) \\
&\quad = (\mathbf{F}, R, Q, X, R(\mathbf{F} \cap X), Q(\mathbf{F}/(\mathbf{F} \cap X)), Rt|_X R^{-1}, Qt|_{(\mathbb{C}^{|\mu^1|+|\mu^2|}/X)} Q^{-1}) \ ,
\end{aligned}$$

q_2, q_3, q_4 are natural projections.

The fiber of q_4 is a Grassmannian of dimension $|\mu^1||\mu^2|$, the fiber of q_3 is isomorphic to $GL(|\mu^1|) \times GL(|\mu_2|)$, the fiber of q_2 is isomorphic to the affine space

$$\bigoplus_i \mathrm{Hom}(\mathbf{F}_i^2/\mathbf{F}_{i-1}^2, \mathbb{C}^{|\mu^1|}/\mathbf{F}_i^1) \ ,$$

and the fiber of q_1 is isomorphic to the affine space

$$\bigoplus_i \operatorname{Hom}(\mathbf{F}_i^2/\mathbf{F}_{i-1}^2, \mathbf{F}_{i-1}^1)\ .$$

Statement 3.8.f follows. □

Statement 3.8.e implies that the set of irreducible components of $\mathfrak{T}'_N(\mathbf{v}, \mu^1, \mu^2)$ is in a natural bijection with $\mathcal{T}_N(\mathbf{v}, \mu^1, \mu^2)$ (the set of irreducible components of $\mathfrak{T}_N(\mathbf{v}, \mu^1, \mu^2)$). On the other hand 3.8.f provides a bijection between the product of the sets of irreducible components of $\mathfrak{M}_N(\mathbf{v}^1, \mu^1)$ and $\mathfrak{M}_N(\mathbf{v}^2, \mu^2)$, and the set of irreducible components of $\mathfrak{T}'_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2)$. Moreover it follows from 3.8.e, 3.8.f, 3.5.a, and 3.8.a that

$$\dim \mathfrak{T}'_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2) = \dim \mathfrak{T}'_N(\mathbf{v}^1 + \mathbf{v}^2, \mu^1, \mu^2)\ .$$

Hence the closures of irreducible components of $\mathfrak{T}'_N(\mathbf{v}^1, \mathbf{v}^2, \mu^1, \mu^2)$ are irreducible components of $\mathfrak{T}'_N(\mathbf{v}^1 + \mathbf{v}^2, \mu^1, \mu^2)$ and one obtains a bijection

$$\mathcal{T}_N(\mathbf{v}, \mu^1, \mu^2) \leftrightarrow \bigsqcup_{\substack{\mathbf{v}^1 \in \mathcal{Q}_N(|\mu^1|) \\ \mathbf{v}^2 \in \mathcal{Q}_N(|\mu^2|) \\ \mathbf{v}^1 + \mathbf{v}^2 = \mathbf{v}}} \mathcal{M}_N(\mathbf{v}^1, \mu^1) \times \mathcal{M}_N(\mathbf{v}^2, \mu^2)\ .$$

Together with (3.8.b) it provides a bijection

$$\tau_N : \bigsqcup_{\substack{\mathbf{v}^1 \in \mathcal{Q}_N(|\mu^1|) \\ \mathbf{v}^2 \in \mathcal{Q}_N(|\mu^2|) \\ \mathbf{v}^1 + \mathbf{v}^2 = \mathbf{v}}} \mathcal{M}_N(\mathbf{v}^1, \mu^1) \times \mathcal{M}_N(\mathbf{v}^2, \mu^2) \xrightarrow{\sim} \bigsqcup_{\lambda \in \mathcal{Q}_N^+(|\mu^1| + |\mu^2|)} \mathcal{S}_2((\mu^1, \mu^2), \lambda) \times \mathcal{M}_N(\mathbf{v}, \lambda)\ ,$$

or, taking union over $\mathbf{v} \in \mathcal{Q}_N(|\mu^1| + |\mu^2|)$,

$$\tau_N : \quad \mathcal{M}_N(\mu^1) \times \mathcal{M}_N(\mu^2) \xrightarrow{\sim} \bigsqcup_{\lambda \in \mathcal{Q}_N^+(|\mu^1| + |\mu^2|)} \mathcal{S}_2((\mu^1, \mu^2), \lambda) \times \mathcal{M}_N(\lambda)\ .$$

3.9. **The main theorem for gl_N.** Here is the main theorem of this paper in the gl_N case.

Theorem. *The map τ_N is a crystal isomorphism*

$$\tau_N : \quad \mathcal{M}_N(\mu^1) \otimes \mathcal{M}_N(\mu^2) \xrightarrow{\sim} \bigoplus_{\lambda \in \mathcal{Q}_N^+(|\mu^1| + |\mu^2|)} \mathcal{S}_2((\mu^1, \mu^2), \lambda) \otimes \mathcal{M}_N(\lambda)\ ,$$

where $\mathcal{S}_2((\mu^1, \mu^2), \lambda)$ is considered as a set with the trivial crystal structure.

The proof of this theorem occupies the next subsection.

3.10. **gl_2-restriction in the tensor product or the proof of Theorem 3.9.** By definition the bijection τ_N preserves the weight function. So in order to prove that τ_N is a crystal morphism

$$\tau_N : \mathcal{M}_N(\mu^1) \otimes \mathcal{M}_N(\mu^2) \to \bigsqcup_{\lambda \in \mathcal{Q}_N^+(|\mu^1| + |\mu^2|)} \mathcal{S}_2((\mu^1, \mu^2), \lambda) \otimes \mathcal{M}_N(\lambda)$$

it remains to compare the action of the Kashiwara operators $\tilde{f}_k$ and $\tilde{e}_k$ on the domain and the range of τ_N (recall that both sides are normal crystals, and thus one does not need to care about the functions ε_k and φ_k). The idea of the proof is to reduce the problem to the gl_2 case and then use the corresponding result for gl_2 (Theorem 2.6).

Let $\mu^1, \mu^2 \in \mathcal{Q}_N^+$, and k be an integer such that $1 \le k \le N-1$. Consider the following variety (cf. 3.6):

$$\begin{aligned}&\mathfrak{T}_N^k(\mu^1,\mu^2)\\ &= \{(t,X,\mathbf{F}) \mid t \in \mathfrak{N}_N(|\mu^1|+|\mu^2|),\ (t,\mathbf{F}) \in \mathfrak{M}_N^k(|\mu^1|+|\mu^2|),\ X \in \mathfrak{S}_2((\mu^1,\mu^2),t)\}\ .\end{aligned}$$

See 3.6 for the definition of $\mathfrak{M}_N^k$. The variety $\mathfrak{T}_N^k(\mu^1,\mu^2)$ represents gl_2-restriction for the tensor product variety $\mathfrak{T}_N(\mu^1,\mu^2)$. One has the following commutative diagram (cf. (3.6.a)):

$$\begin{tikzcd} & \mathfrak{T}_N(\mu^1,\mu^2) \arrow[dl, "\xi_N^k"'] \arrow[dd, "\delta_N"] \\ \mathfrak{T}_N^k(\mu^1,\mu^2) \arrow[dr, "\delta_N^k"'] & \\ & \mathfrak{N}_N(|\mu^1|+|\mu^2|) \end{tikzcd}$$

where ξ_N^k forgets the k-th subspace in the flag $\mathbf{F}$ and δ_N^k forgets the rest of the flag. Given $\mathbf{u} \in \mathcal{Q}_N^k(|\mu^1|+|\mu^2|)$, and $r \in \mathbb{Z}_{\ge 0}$ consider the following stratum in $\mathfrak{T}_N^k(\mu^1,\mu^2)$:

$$\mathfrak{T}_N^k(\mathbf{u},\mu^1,\mu^2,r) = \{(t,X,\mathbf{F}) \in \mathfrak{T}_N^k(\mu^1,\mu^2) \mid \dim \mathbf{F} = \mathbf{u},\ \operatorname{rank}(t|_{\mathbf{F}_{k+1}/\mathbf{F}_{k-1}}) = r\}\ .$$

The fiber of ξ_N^k over a point in $\mathfrak{T}_N^k(\mathbf{u},\mu^1,\mu^2,r)$ is isomorphic to $\mathfrak{M}_2(\mathbf{u}_{k+1},r)$ (cf. 2.2). Consider stratification of $\mathfrak{N}_N(|\mu^1|+|\mu^2|)$ by $GL(|\mu^1|+|\mu^2|)$-orbits $\{O_\lambda\}$. The map δ_N^k restricted to $(\delta_N^k)^{-1}(O_\lambda)$ is a locally trivial fibration with a fiber isomorphic to the product of two Spaltenstein varieties $\mathfrak{S}_2((\mu^1,\mu^2),\lambda)$ and $\mathfrak{M}_N^k(\mathbf{u},\lambda,r)$ (cf. 3.6). Hence $\mathfrak{T}_N^k(\mathbf{u},\mu^1,\mu^2,r)$ is of pure dimension,

$$\begin{aligned}\text{(3.10.a)}\quad \dim \mathfrak{T}_N^k(\mathbf{u},\mu^1,\mu^2,r) &= |\mu^1||\mu^2| \\ &+ \frac{1}{2}\Big(\sum_{i\ne j}\mathbf{u}_i\mathbf{u}_j + \dim O_{\mu^1} + \dim O_{\mu^2} + \dim O_{(\mathbf{u}_{k+1}-r,r)}\Big)\ ,\end{aligned}$$

and one has the following bijection

$$\text{(3.10.b)}\qquad \mathcal{T}_N^k(\mathbf{u},\mu^1,\mu^2,r) \leftrightarrow \bigsqcup_{\lambda \in \mathcal{Q}_N^+(|\mu^1|+|\mu^2|)} \mathcal{S}_2((\mu^1,\mu^2),\lambda) \times \mathcal{M}_N^k(\mathbf{u},\lambda,r)\ ,$$

where $\mathcal{T}_N^k(\mathbf{u},\mu^1,\mu^2,r)$ is the set of irreducible components of $\mathfrak{T}_N^k(\mathbf{u},\mu^1,\mu^2,r)$.

Recall that the fiber of ξ_N^k over a point in $\mathfrak{T}_N^k(\mathbf{u},\mu^1,\mu^2,r)$ is isomorphic to $\mathfrak{M}_2(\mathbf{u}_{k+1},r)$. Since

$$\dim \mathfrak{T}_N^k(\rho_N^k(\mathbf{v}),\mu^1,\mu^2,r) + \dim \mathfrak{M}_2(\mathbf{v}_k,\mathbf{v}_k+\mathbf{v}_{k+1},r) = \dim \mathfrak{T}_N(\mathbf{v},\mu^1,\mu^2)$$

one has a bijection

$$\text{(3.10.c)}\qquad \mathcal{T}_N(\mu^1,\mu^2) \leftrightarrow \bigsqcup_{\substack{\mathbf{u}\in\mathcal{Q}_N^k(|\mu^1|+|\mu^2|)\\ r\in\mathbb{Z}_{\ge 0}}} \mathcal{T}_N^k(\mathbf{u},\mu^1,\mu^2,r) \times \mathcal{M}_2(\mathbf{u}_{k+1},r)\ .$$

Combining (3.10.b) with (3.10.c) one obtains a bijection

(3.10.d)
$$\mathcal{T}_N(\mu^1,\mu^2) \leftrightarrow \bigsqcup_{\substack{\lambda\in\mathcal{Q}_N^+(|\mu^1|+|\mu^2|)\\ \mathbf{u}\in\mathcal{Q}_N^k(|\mu^1|+|\mu^2|)\\ r\in\mathbb{Z}_{\geq 0}}} \mathcal{S}_2((\mu^1,\mu^2),\lambda)\times\mathcal{M}_N^k(\mathbf{u},\lambda,r)\times\mathcal{M}_2(\mathbf{u}_{k+1},r)\ .$$

This bijection is the result of gl_2-restriction on the range of τ_N. In particular, the operators $\tilde{e}_k$ and $\tilde{f}_k$ act on the last multiple in the RHS of (3.10.d).

To carry out gl_2-restriction on the domain of τ_N one has to forget the k-th subspace of the flag $\mathbf{F}$ in all varieties in the diagrams 3.8.c and 3.8.d. The results are the following commutative diagrams

(3.10.e)
$$\begin{tikzcd}
\mathfrak{T}_N(\mathbf{v},\mu^1,\mu^2) \arrow[d, "\xi_N^k"'] & \mathfrak{T}'_N(\mathbf{v},\mu^1,\mu^2) \arrow[l, "a_N"'] \arrow[r, dotted, "b_N"] \arrow[d, "\xi'^k_N"'] & \mathfrak{M}'_N(\mu^1)\times\mathfrak{M}'_N(\mu^2) \arrow[d, "\sigma_N^k\times\sigma_N^k"'] \\
\mathfrak{T}_N^k(\rho_N^k(\mathbf{v}),\mu^1,\mu^2) & \mathfrak{T}'^k_N(\rho_N^k(\mathbf{v}),\mu^1,\mu^2) \arrow[l, "c_N^k"'] \arrow[r, dotted, "d_N^k"] & \mathfrak{M}'^k_N(\mu^1)\times\mathfrak{M}'^k_N(\mu^2) \arrow[d, "\pi_N^k\times\pi_N^k"'] \\
 & & O_{\mu^1}\times O_{\mu^2}
\end{tikzcd}$$

and

(3.10.f)
$$\begin{tikzcd}
\mathfrak{T}_N(\mathbf{v}^1,\mathbf{v}^2,\mu^1,\mu^2) \arrow[d, "\xi_N^k"'] & \mathfrak{T}'_N(\mathbf{v}^1,\mathbf{v}^2,\mu^1,\mu^2) \arrow[l, "a_N"'] \arrow[r, "b_N"] \arrow[d, "\xi'^k_N"'] & \mathfrak{M}'_N(\mathbf{v}^1,\mu^1)\times\mathfrak{M}'_N(\mathbf{v}^2,\mu^2) \arrow[d, "\sigma_N^k\times\sigma_N^k"'] \\
\mathfrak{T}_N^k(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2) & \mathfrak{T}'^k_N(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2) \arrow[l, "c_N^k"'] \arrow[r, "d_N^k"] & \mathfrak{M}'^k_N(\mathbf{u}^1,\mu^1)\times\mathfrak{M}'^k_N(\mathbf{u}^2,\mu^2) \arrow[d, "\pi_N^k\times\pi_N^k"'] \\
 & & O_{\mu^1}\times O_{\mu^2}
\end{tikzcd}$$

where $\mathbf{u}^1=\rho_N^k(\mathbf{v}^1)$, $\mathbf{u}^2=\rho_N^k(\mathbf{v}^2)$. In the above two diagrams the top rows are copies of the top rows of diagrams 3.8.c and 3.8.d;

$\mathfrak{T}_N^k(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2)$
$\quad = \{(t,X,\mathbf{F})\in\mathfrak{T}_N^k(\mu^1,\mu^2) \mid \dim(\mathbf{F}\cap X)=\mathbf{u}^1,\ \dim(\mathbf{F}/(\mathbf{F}\cap X))=\mathbf{u}^2\}$;

$\mathfrak{T}'^k_N(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2)$ (resp. $\mathfrak{T}'^k_N(\mathbf{u},\mu^1,\mu^2)$) is the variety of triples (x,R,Q), where $x=(t,X,\mathbf{F})\in\mathfrak{T}_N^k(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2)$ (resp. $x=(t,X,\mathbf{F})\in\mathfrak{T}_N^k(\mathbf{u},\mu^1,\mu^2)$), and R (resp. Q) is an isomorphism $X\to\mathbb{C}^{|\mu^1|}$ (resp. $\mathbb{C}^{|\mu^1|+|\mu^2|}/X\to\mathbb{C}^{|\mu^2|}$);

the vertical maps are natural projections (forgetting the k-th subspace of the flag $\mathbf{F}$);

the maps c_N^k and d_N^k are uniquely defined to make the diagrams commutative. The map d_N^k is regular only in the diagram (3.10.f). Let

$$\begin{aligned}
&\mathfrak{T}_N^k(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2,r^1,r^2,r)\\
&\qquad = \{(t,X,\mathbf{F})\in\mathfrak{T}_N^k(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2) \mid \operatorname{rank}(t|_{\mathbf{F}_{k+1}/\mathbf{F}_{k-1}})=r,\\
&\qquad\quad \operatorname{rank}(t|_{(X\cap\mathbf{F}_{k+1})/(X\cap\mathbf{F}_{k-1})})=r^1,\ \operatorname{rank}(t|_{(\mathbf{F}_{k+1}/((\mathbf{F}_{k+1}\cap X)+\mathbf{F}_{k-1})})=r^2\}\ ,
\end{aligned}$$

and

$$\mathfrak{T}'^k_N(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2,r^1,r^2,r) = (c^k_N)^{-1}(\mathfrak{T}^k_N(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2,r^1,r^2,r)) \;.$$

The set $\mathfrak{T}^k_N(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2,r^1,r^2,r)$ (resp. $\mathfrak{T}'^k_N(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2,r^1,r^2,r)$) is a locally closed subset in $\mathfrak{T}^k_N(\mathbf{u}^1+\mathbf{u}^2,\mu^1,\mu^2,r)$ (resp. $\mathfrak{T}'^k_N(\mathbf{u}^1+\mathbf{u}^2,\mu^1,\mu^2,r)$).

The proof of the following proposition is similar to the proofs of Propositions 2.5 and 3.8.

Proposition.

3.10.g. *in the diagrams* (3.10.e) *and* (3.10.f) *the maps* a_N *and* c^k_N *are principal* $GL(|\mu^1|)\times GL(|\mu^2|)$*-bundles;*

3.10.h. *if* $r^1+r^2 \le r \le \min(\mathbf{u}^2_{k+1}-r^2+r^1, \mathbf{u}^1_{k+1}-r^1+r^2)$*, then the map* d^k_N *restricted to* $\mathfrak{T}'^k_N(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2,r^1,r^2,r)$ *is a locally trivial fibration over* $\mathfrak{M}'^k_N(\mathbf{u}^1,\mu^1,r^1)\times\mathfrak{M}'^k_N(\mathbf{u}^2,\mu^2,r^2)$ *with a constant smooth connected fiber of dimension*

$$\dim GL(|\mu^1|)+\dim GL(|\mu^2|)+|\mu^1||\mu^2|+\sum_{i\neq j}\mathbf{u}^1_i\mathbf{u}^2_j$$
$$+\frac{1}{2}(\dim O_{(\mathbf{u}^1_{k+1}+\mathbf{u}^2_{k+1}-r,r)}-\dim O_{(\mathbf{u}^1_{k+1}-r^1,r^1)}-\dim O_{(\mathbf{u}^2_{k+1}-r^2,r^2)})\;,$$

otherwise $\mathfrak{T}'^k_N(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2,r^1,r^2,r)$ *is empty.*

It follows from 3.10.g and 3.10.a that the variety $\mathfrak{T}'^k_N(\mathbf{u},\mu^1,\mu^2,r)$ has pure dimension,

$$\text{(3.10.i)}\quad \dim\mathfrak{T}'^k_N(\mathbf{u},\mu^1,\mu^2,r)=\dim GL(|\mu^1|)+\dim GL(|\mu^2|)+|\mu^1||\mu^2| + \frac{1}{2}(\sum_{i\neq j}\mathbf{u}_i\mathbf{u}_j+\dim O_{\mu^1}+\dim O_{\mu^2}+\dim O_{(\mathbf{u}_{k+1}-r,r)})\;,$$

and the set of irreducible components of $\mathfrak{T}'^k_N(\mathbf{u},\mu^1,\mu^2,r)$ is in a natural bijection with $\mathcal{T}^k_N(\mathbf{u},\mu^1,\mu^2,r)$ (the set of irreducible components of $\mathfrak{T}^k_N(\mathbf{u},\mu^1,\mu^2,r)$).

On the other hand it follows from 3.10.h, (3.6.b), and (3.10.i) that the variety $\mathfrak{T}'^k_N(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2,r^1,r^2,r)$ has the same dimension as $\mathfrak{T}'^k_N(\mathbf{u}^1+\mathbf{u}^2,\mu^1,\mu^2,r)$ (in particular, the dimension does not depend on r^1, r^2, and $\mathbf{u}^1-\mathbf{u}^2$). Hence irreducible components of $\mathfrak{T}'^k_N(\mathbf{u}^1+\mathbf{u}^2,\mu^1,\mu^2,r)$ are closures of irreducible components of $\mathfrak{T}'^k_N(\mathbf{u}^1,\mathbf{u}^2,\mu^1,\mu^2,r^1,r^2,r)$ and using 3.10.h one obtains the following bijection:

$$\text{(3.10.j)}\quad \mathcal{T}^k_N(\mathbf{u},\mu^1,\mu^2,r)\leftrightarrow \bigsqcup_{\substack{\mathbf{u}^1\in\mathcal{Q}^k_N(|\mu^1|)\\ \mathbf{u}^2\in\mathcal{Q}^k_N(|\mu^2|)\\ \mathbf{u}^1+\mathbf{u}^2=\mathbf{u}\\ r^1,\, r^2\in\mathbb{Z}_{\ge 0}}} \mathcal{S}_2(((\mathbf{u}^1_{k+1},r^1),(\mathbf{u}^2_{k+1},r^2)),(\mathbf{u}_{k+1},r)) \times\mathcal{M}^k_N(\mathbf{u}^1,\mu^1,r^1)\times\mathcal{M}^k_N(\mathbf{u}^2,\mu^2,r^2)$$

where $\mathcal{S}_2(((\mathbf{u}^1_{k+1},r^1),(\mathbf{u}^2_{k+1},r^2)),(\mathbf{u}_{k+1},r))$ comes from the fiber of d^k_N (cf. the inequality in 3.10.h).

Combining (3.10.j) with (3.10.b) one obtains a bijection

(3.10.k)
$$\tau^k_N : \bigsqcup_{\substack{\mathbf{u}^1\in\mathcal{Q}^k_N(|\mu^1|)\\ \mathbf{u}^2\in\mathcal{Q}^k_N(|\mu^2|)\\ \mathbf{u}^1+\mathbf{u}^2=\mathbf{u}\\ r^1,\, r^2\in\mathbb{Z}_{\geq 0}}} \mathcal{S}_2(((\mathbf{u}^1_{k+1}, r^1), (\mathbf{u}^2_{k+1}, r^2)), (\mathbf{u}_{k+1}, r)) \times \mathcal{M}^k_N(\mathbf{u}^1, \mu^1, r^1) \times \mathcal{M}^k_N(\mathbf{u}^2, \mu^2, r^2)$$
$$\xrightarrow{\sim} \bigsqcup_{\lambda\in\mathcal{Q}^+_N(|\mu^1|+|\mu^2|)} \mathcal{S}_2((\mu^1, \mu^2), \lambda) \times \mathcal{M}^k_N(\mathbf{u}, \lambda, r)\ .$$

Let $\mu^1,\ \mu^2 \in \mathcal{Q}^+_N$, and $\mathbf{v} \in \mathcal{Q}_N(|\mu^1| + |\mu^2|)$. Consider the following diagram of bijections:

(3.10.l)
$$\begin{tikzcd}
\displaystyle\bigsqcup_{\lambda\in\mathcal{Q}^+_N(|\mu^1|+|\mu^2|)} \mathcal{S}_2((\mu^1, \mu^2), \lambda) \times \mathcal{M}_N(\mathbf{v}, \lambda) \arrow[dddd, bend left=60, "\mathrm{Id}\times\theta^k_N"] \\
\displaystyle\bigsqcup_{\substack{\mathbf{v}^1\in\mathcal{Q}_N(|\mu^1|)\\ \mathbf{v}^2\in\mathcal{Q}_N(|\mu^2|)\\ \mathbf{v}^1+\mathbf{v}^2=\mathbf{v}}} \mathcal{M}_N(\mathbf{v}^1, \mu^1) \times \mathcal{M}_N(\mathbf{v}^2, \mu^2) \arrow[u, "\tau_N"'] \arrow[d, "\theta^k_N\times\theta^k_N"] \\
\displaystyle\bigsqcup_{\substack{\mathbf{v}^1\in\mathcal{Q}_N(|\mu^1|)\\ \mathbf{v}^2\in\mathcal{Q}_N(|\mu^2|)\\ \mathbf{v}^1+\mathbf{v}^2=\mathbf{v}\\ r^1,\, r^2\in\mathbb{Z}_{\geq 0}}} \begin{array}{c}\mathcal{M}^k_N(\rho^k_N(\mathbf{v}^1), \mu^1, r^1) \times \mathcal{M}_2(\mathbf{v}^1_k, (\rho^k_N(\mathbf{v}^1))_{k+1}, r^1)\times\\ \mathcal{M}^k_N(\rho^k_N(\mathbf{v}^2), \mu^2, r^2) \times \mathcal{M}_2(\mathbf{v}^2_k, (\rho^k_N(\mathbf{v}^2))_{k+1}, r^2)\end{array} \arrow[d, "(\mathrm{Id}\times\mathrm{Id}\times\tau_2)\circ P_{23}"] \\
\displaystyle\bigsqcup_{\substack{\mathbf{u}^1\in\mathcal{Q}^k_N(|\mu^1|)\\ \mathbf{u}^2\in\mathcal{Q}^k_N(|\mu^2|)\\ \mathbf{u}^1+\mathbf{u}^2=\rho^k_N(\mathbf{v})\\ r^1,\, r^2,\, r\in\mathbb{Z}_{\geq 0}}} \begin{array}{c}\mathcal{M}^k_N(\mathbf{u}^1, \mu^1, r^1) \times \mathcal{M}^k_N(\mathbf{u}^2, \mu^2, r^2)\times\\ \mathcal{S}_2(((\mathbf{u}^1_{k+1}, r^1), (\mathbf{u}^2_{k+1}, r^2)), ((\rho^k_N(\mathbf{v}))_{k+1}, r))\times\\ \mathcal{M}_2(\mathbf{v}_k, (\rho^k_N(\mathbf{v}))_{k+1}, r)\end{array} \arrow[d, "\tau^k_N\times\mathrm{Id}"] \\
\displaystyle\bigsqcup_{\substack{\lambda\in\mathcal{Q}^+_N(|\mu^1|+|\mu^2|)\\ r\in\mathbb{Z}_{\geq 0}}} \begin{array}{c}\mathcal{S}_2((\mu^1, \mu^2), \lambda) \times \mathcal{M}^k_N(\rho^k_N(\mathbf{v}), \lambda, r)\times\\ \mathcal{M}_2(\mathbf{v}_k, (\rho^k_N(\mathbf{v}))_{k+1}, r)\end{array}
\end{tikzcd}$$

where P_{23} is the permutation of the second and the third multiples in the direct product. It follows from constructions of the bijections τ and θ (or, more precisely, from comparison of the right squares in the diagrams (3.10.f) and (2.5.b)) that the diagram (3.10.l) is commutative.

The commutativity of the diagram (3.10.l) together with Theorem 2.6 implies that the bijection τ_N commutes with the action of the Kashiwara operators $\tilde{e}_k$ and $\tilde{f}_k$. Therefore τ_N is a morphism of crystals

$$\tau_N : \mathcal{M}_N(\mu^1) \otimes \mathcal{M}_N(\mu^2) \xrightarrow{\sim} \bigsqcup_{\lambda\in\mathcal{Q}^+_N(|\mu^1|+|\mu^2|)} \mathcal{S}_2((\mu^1, \mu^2), \lambda) \otimes \mathcal{M}_N(\lambda)\ .$$

This concludes the proof of Theorem 3.9.

3.11. **Corollary.** The following proposition is a corollary of Theorem 3.9.

Proposition. *Let μ, μ^1, and $\mu^2 \in \mathcal{Q}_N^+$. Then*

3.11.a. *the crystal $\mathcal{M}_N(\mu)$ is isomorphic to $\mathcal{L}(\mu)$ (the crystal of the canonical basis of a highest weight irreducible representation with the highest weight μ);*

3.11.b. *the cardinal of the set $\mathcal{S}_2((\mu^1, \mu^2), \lambda)$ is equal to the Littlewood-Richardson coefficient $c^\lambda_{\mu^1\mu^2}$.*

Proof. The proposition follows from Theorem 3.9, Theorem 1.3, and Proposition 3.7. □

Remark. Note that 3.11.b is the Hall Theorem (cf. 3.3). Statement 3.11.a could be deduced from the results of Ginzburg concerning gl_N-action in the top homology of the variety $\mathfrak{M}_N(\mu)$ (cf. [Gin91]).

3.12. **Multiple tensor product and Levi restriction.** There are two straightforward generalizations of the constructions of this section.

First, one can consider a more general tensor product variety $\mathfrak{T}_N(\mu^1, \dots, \mu^l)$, which corresponds to a product of l polynomial representations of gl_N. A point of this variety is a triple $(t, \mathbf{X}, \mathbf{F})$ consisting of a nilpotent operator t, an l-step partial flag $\mathbf{X}$, and an N-step partial flag $\mathbf{F}$, such that t preserves both $\mathbf{X}$ and $\mathbf{F}$ and when restricted to the subfactors of $\mathbf{X}$ (resp. $\mathbf{F}$) gives operators with Jordan forms $\mu^1, \dots, \mu^l$ (resp. 0 operators). In this way one can prove a generalization of the Hall theorem, saying that the number of irreducible components of the Spaltenstein variety $\mathfrak{S}_l((\mu^1, \dots, \mu^l), \lambda)$ is equal to the multiplicity of the representation $L(\lambda)$ in $L(\mu^1) \otimes \dots \otimes L(\mu^l)$.

Second, one can generalize the notion of gl_2-restriction. Namely forgetting several subspaces in the flag $\mathbf{F}$ one can define restriction to a Levi subalgebra of a parabolic subalgebra of gl_N. In this way one would obtain a bijection similar to τ_N^k (cf. (3.10.k)) relating tensor products for gl_N and for the Levi subalgebra.

References

[BM83] W. Borho and R. MacPherson, *Partial resolutions of nilpotent varieties*, Astérisque, **101-102** (1983), pp. 23–74. MR **85j:**14087

[Gin91] V. Ginzburg, *Lagrangian construction of the enveloping algebra* $U(\mathrm{sl}_n)$, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 12, 907–912. MR **92c:**17017

[GL92] I. Grojnowski and G. Lusztig, *On bases of irreducible representations of quantum* gl_n, Kazhdan-Lusztig theory and related topics, Contemp. Math. **139** (Chicago, IL, 1989), AMS, Providence, RI, 1992, pp. 167–174. MR **94a:**20070

[Hal59] P. Hall, *The algebra of partitions*, Proc. 4th Canadian Math. Congress (Banff), University of Toronto Press, 1959, pp. 147–159. MR **28:**1074

[Jos95] A. Joseph, *Quantum groups and their primitive ideals*, Springer-Verlag, Berlin, 1995. MR **96d:**17015

[Kas90] M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Comm. Math. Phys. **133** (1990), no. 2, 249–260.

[Kas91] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516. MR **93b:**17045

[Kas94] M. Kashiwara, *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. **73** (1994), no. 2, 383–413. MR **95c:**17024

[KS97] M. Kashiwara and Y. Saito, *Geometric construction of crystal bases*, Duke Math. J. **89** (1997), no. 1, 9–36. MR **99e:**17025

[Lus91] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), no. 2, 365–421. MR **91m:**17018

[LW54] S. Lang and A. Weil, *Number of points of varieties in finite fields*, Amer. J. Math. **76** (1954), 819–827. MR **16:**398d

[Mac95] I. Macdonald, *Symmetric functions and Hall polynomials*, second ed., The Clarendon Press Oxford University Press, New York, 1995. MR **96h:**05207

[Mal01] A. Malkin, *Tensor product varieties and crystals. ADE case.*, arXiv:math.AG/0103025

[Nak94] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.

[Nak98] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560. MR **99b:**17033

[Nak01] H. Nakajima, *Quiver varieties and tensor products*, arXiv:math.QA/0103008

[Spa82] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Springer-Verlag, Berlin, 1982. MR **84a:**14024

Department of Mathematics, Yale University, P.O. Box 208283, New Haven, Connecticut 06520-8283

Current address: Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139-4307

E-mail address: `malkin@math.mit.edu`